Applied Analysis

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In this course we will introduce some basic mathematical theory about calculus of variations and optimal control, which has wide applications in many areas such as in physics, engineering and economics. The name of calculus of variations is given to the theory of optimizing integrals in mid-eighteenth century, but such optimization problems have long existed in the history of mathematics. For example, the famous *Brachistochrone* problem posted by John Bernoulli in 1696 asks one to find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip from one point to another in the least time. On the other hand, optimal control is an area developed since the 1950s when U.S. and Russia competed to explore the solar system, where we wish to construct trajectories along which a space vehicle will reach its destination in minimum amount of time or consuming minimum amount of fuel. As we will see, such problems may be solved by using ideas from the theory of calculus of variations.

More specifically, in this course we will start by reviewing/introducing some necessary backgrounds from calculus and ordinary differential equations (ODEs), such as the Taylor series expansion and some basic solution methods for ODEs. To motivate the ideas in calculus of variations, we first consider optimizing functions of single and multivariables, which can be done from calculus via derivative tests and the method of Lagrange multipliers. Then we try to generalize the situation to optimizing integrals where we use the variational methods to derive the *Euler-Lagrange* equation, which is the central result of the calculus of variations. Finally we consider some basic optimal control problems and see how ideas from calculus of variations may be applied in there. In particular, we will introduce the celebrated *maximum principle* of Pontryagin.