Algebraic combinatorics II final project problems

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1 Trace function of induced realizations

Let G be a finite group, and S be a subgroup of G. Let C denote a conjugacy class of G. So $C \cap S$ decomposes into conjugacy classes $C_1, ..., C_r$ of S. Let W be a realization of S. Consider Ind W, i.e. the corresponding induced realization on G.

1. Prove that

$$Trace_{\text{Ind } W}(C) = \frac{|G|}{|S|} \sum_{i=1}^{r} \frac{|C_i|}{|C|} Trace_W(C_i).$$

.

2. Can you think of an application of this formula? i.e. Can you use this formula to derive some useful information in a suitable setup?

2 Multiplication to one problem

Let C_i denote a conjugacy class in $Perm(n)$ that consists of permutations that have i_1 1-cycles, i_2 2-cycles,..., and i_n n-cycles.

Let $x_1, ..., x_k$ be independent variables. Define $P_j(x) = x_1^j + x_2^j + ... + x_k^j$, and $\Delta(x) = \prod_{i<j}(x_i - x_j)$.

Given a partition λ of n: $\lambda_1 \geq ... \geq \lambda_k \geq 0$. Define $l_1 = \lambda_1 + k - 1$, $l_2 = \lambda_2 + k - 2$,..., $l_k = \lambda_k$. Assume the following formula for the trace function of the minimal realization of $Perm(n)$ corresponding to λ:

$$Trace_\lambda(C_i) = \text{The coefficient of } x_1^{l_1}...x_k^{l_k} \text{ in } \Delta(x)\prod_{j=1}^{n} P_j(x)^{i_j}.$$

1. Let g be a cycle of length n in $Perm(n)$. Try to guess a formula for $Trace_\lambda(g)$ for the minimal realization of $Perm(n)$ corresponding to any given partition λ of n. Try to prove your formula.

2. Let g_1, g_2, g_3 be 3 random cycles of length n in $Perm(n)$, each selected with uniform probability distribution. Find the probability that $g_1 g_2 g_3 = e$.

3 Graphs of realizations

Let $G_0 = \left\{2 \times 2 \text{ matrix } A \text{ whose entries are complex numbers} \mid \overline{A}^t A = I, \det(A) = 1\right\}$. Clearly G_0 is a group under matrix multiplication, and G_0 has a 2-dimensional realization over $\mathbb{C}$ by multiplying on length 2 column vectors.

Let H be a finite subgroup of G_0. H has a 2-dimensional realization over $\mathbb{C}$ via its embedding in G_0, which we denote by W. Let $V_1, ..., V_k$ denote all the isomorphism classes of finite dimensional minimal realizations of H.

1. Show that if V_i appears a_{ij} many times in $W \otimes V_j$, then V_j appears a_{ij} many times in $W \otimes V_i$.

2. We define a graph G as follows: its vertex set is in bijection with the isomorphism classes of finite dimensional irreducible realizations of H, i.e. the $V_1, ..., V_k$, and V_i and V_j are connected by a_{ij} many edges with the a_{ij} as defined above.

   (a) Show that $a_{ij} \leq 1$ for all i, j.

   (b) If G is a cycle, what can you say about H?

   (c) What possible graphs can arise in this way?

4 Invariants of a realization

Let $G = \{n \times n \text{ matrix } A \text{ whose entries are in the finite field } \mathbb{F}_2 \mid \det(A) = 1\}$. Then G is a finite group under multiplication.

Multiplying on length n column vectors gives a realization ρ of G on a n-dimensional vector space V over $\mathbb{F}_2$ with standard basis $< x_1, ..., x_n >$. Now let W denote the vector space over $\mathbb{F}_2$ spanned by the basis given by all monomials in $x_1, ..., x_n$ of degree n. Then ρ naturally gives rise to a realization ρ_W of G on W. Let ρ_W^* denote the dual realization on the dual space W^*. Find a vector a in W^* that is an invariant under the ρ_W^* realization of G. i.e. a vector a such that $\rho_W^*(g)(a) = a$ for any $g \in G$. Prove your assertion.