LINEAR ALGEBRA HOMEWORK 4 – WEDNESDAY 8/1

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Assume U, V, W are F-vector spaces. Put $\operatorname{End} V = \operatorname{Hom}(V, V)$.

Exercise 0.1. *Find the dimension of $M_{2,2}$ by giving a basis of this vector space. Generalize your result to $M_{k,l}$.*

Exercise 0.2. *Let $f, g : V \to V$ be two given maps such that $f \circ g = id_V$.*
(a) Show that g is injective and f is surjective.
(b) Assume in addition that $\dim V < +\infty$ and f is linear. Show that f is injective, hence g is surjective. (Hint: Use COD.)
(c) Conclude that g is bijective, and that $f = g^{-1}$ and $g \circ f = id_V$.
(d) Let $A, B \in M_{n,n}$. Show that if $AB = I$, then $BA = I$.

Exercise 0.3. *Another proof. Show that if $\ker(BA) = (0)$ then $\ker A = (0)$, hence A is an isomorphism. Conclude that $B = A^{-1}$. (Hint: COD.)*

Exercise 0.4. *WRITE UP (Transpose) Recall the* **transpose** *operation*

$$^T : M_{2\times 2} \to M_{2\times 2}, \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}.$$

(a) Show that $(AB)^T = B^T A^T$ for $A, B \in M_{2,2}$.
(b) Prove that A is invertible iff its A^T is invertible. Moreover, in this case $(A^T)^{-1} = (A^{-1})^T$.
(c) Prove that A is invertible iff its rows form a basis of F^2.
(Hint: Use (a) and the MTC.)
Generalize this.

Exercise 0.5. *Decide if $A = [e_3, e_1+e_2, e_2] \in M_{3,3}$ is invertible. If so, compute A^{-1}.*

Exercise 0.6. *WRITE UP Let Aut_n be the set of all $n \times n$ invertible matrices over F. For $x \in M_{n,n}$, define*

$$[x] := \{\Phi_g(x) = g^{-1}xg \mid g \in \operatorname{Aut}_n\}$$

called the **conjugation class** *in $M_{n,n}$ represented by x.*
(a) Show that two conjugation classes are either equal or disjoint. Therefore, every $x \in M_{n,n}$ lies in exactly one conjugation class, namely $[x]$.
(b) Show that any two matrices in the same class $[x]$ are conjugates of one another, i.e. if $x', x'' \in [x]$ then $x'' = g^{-1}x'g$ for some $g \in \operatorname{Aut}_n$.

Exercise 0.7. *We did this in class, but make sure you can do it too. Let $U \subset V$ be a subspace and $x \in \operatorname{End} V$ such that $xU \subset U$. In 5 lines, prove that the induced map*

$$\bar{x} : V/U \to V/U, \quad v + U \mapsto xv + U$$

satisfies the following: if $p(t) \in F[t]$*, and* $p(x) = 0$ *in* $\operatorname{End} V$ *then* $p(\bar{x}) = 0$ *in* $\operatorname{End} V/U$*.*

Exercise 0.8. *By row reduction, compute*

$$\det[e_3 + e_2 + e_1, e_1 + e_2, e_2].$$

Redo this it by using linearity of $\det$ *in each column.*

Exercise 0.9. *Assume that* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}$ *is invertible. Find a formula for* A^{-1}*. That is to say, find each entry of* A^{-1} *in terms of the 4 entries* a_{ij} *of* A*. Be sure to check that you do get* $AA^{-1} = A^{-1}A = I$*. From this, can you guess the answer for* 3×3 *matrices.*