1. What is Diophantine analysis? (15 July 2019).

1.Two Dirichlet theorems.
a. **Theorem 1.** *For any* $\alpha \in \mathbb{R}$ *and for any* $Q \in \mathbb{Z}_+$ *there exists* $q \in \mathbb{Z}_+$ *satisfying*

$$1 \le q \le Q, \quad ||q\alpha|| \le \frac{1}{Q}, \qquad ||x|| = \min_{a\in\mathbb{Z}} |x - a| \ - \textit{ distance to the nearest integer}$$

b. **Theorem 2.** *For any* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *there exist infinitely many rational fractions* $\frac{p}{q}$ *such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

2. Optimality. For any fraction $\frac{p}{q}$ one has

$$\left|\sqrt{2} - \frac{p}{q}\right| \ge \frac{c}{q^2}$$

with some positive constant c.

3. Algebraic numbers. A number $\alpha \in \mathbb{C}$ is called algebraic if there exists a non-zero polynomial $P(x) \in \mathbb{Q}[x]$ such that $P(\alpha) = 0$.
a. Do there exist (real) numbers which are not algebraic?
b. **Theorem.** *For any algebraic number* α *there exists the unique* **minimal polynomial** $P_\alpha(x)$ *satisfying*

1) $P_\alpha(x) \in \mathbb{Q}[x]$*;*
2) $P_\alpha(\alpha) = 0$*;*
3) the leading coefficient of $P_\alpha(x)$ *is equal to 1.*
4) $P_\alpha(x)$ *has minimal degree among all the polynomials satisfying 1), 2), 3).*

The degree $\deg \alpha$ of an algebraic number α is defined as the degree of the polynomial $P_\alpha(x)$.

4. a. **Liouville theorem.** *Let* α *be an algebraic number of degree* $n = \deg \alpha \ge 2$ *Then there exists positive* c_α *such that*

$$\left|\alpha - \frac{p}{q}\right| \ge \frac{c_\alpha}{q^n} \quad \forall \frac{p}{q}.$$

b. How one can formulate this theorem for algebraic numbers of degree 1?

5. Liouville numbers. Prove that the number

$$\sum_{n=0}^{\infty} \frac{1}{2^{n!}}$$

is transcendental (not algebraic)

6. Some history: Thue-Siegel-Roth theorem. (We will not prove it.)*Let* α *be an algebraic number of degree* $n = \deg \alpha \ge 2$. *Then for any* $\varepsilon > 0$ *there exists positive* $c_{\alpha,\varepsilon}$ *such that*

$$\left|\alpha - \frac{p}{q}\right| \ge \frac{c_{\alpha,\varepsilon}}{q^\gamma} \quad \forall \frac{p}{q} \in \mathbb{Q},$$

where A. Thue: $\gamma = \frac{n}{2} + 1 + \varepsilon$;   C. Roth: $\gamma = 2 + \varepsilon$.

S. Lang's conjecture: $\exists c, \beta$ such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^2(\log q)^\beta} \quad \forall \frac{p}{q}.$$

7. Number e.

Theorem. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ *is not algebraic.*

We do not prove this theorem, the exercise will be to prove that $e \notin \mathbb{Q}$.

Exercises.

1. "Very precise" Dirichlet theorem.

a. For any $Q \in \mathbb{Z}_+$ there exists $q \in \mathbb{Z}_+$ such that $||q\alpha|| \leq \frac{1}{Q+1}$, $q \leq Q$;

b. for any $\tau \geq 1$ there exists q such that $||q\alpha|| < \frac{1}{\tau}$, $q \leq \tau$;

c. for any $\tau \geq 1$ there exists an irreducible fraction $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\tau q}, \quad 1 \leq q \leq \tau.$$

2. Simultaneous approximation. (If somebody is afraid he can assump that $n = 2$.)

a. Let $\alpha_1, ..., \alpha_n$ be real numbers and $Q \in \mathbb{Z}_+$. Prove that

$$\min_{q \in \mathbb{Z}_+,\, q \leq Q^n} \max_{1 \leq j \leq n} ||q\alpha_j|| \leq \frac{1}{Q}.$$

a. Let $\alpha_1, ..., \alpha_n$ be real numbers and not all of them are rational. Prove that there exists infinitely many $q \in \mathbb{Z}_+$ such that

$$\max_{1 \leq j \leq n} ||q\alpha_j|| \leq \frac{1}{q^{1/n}}.$$

3. Linear form. (If somebody is afraid he can suppose that $n = 2$.)

a. Let $\alpha_1, ..., \alpha_n$ be real numbers and $Q \in \mathbb{Z}_+$. Prove that

$$\min_{q_1,...,q_n \in \mathbb{Z}, 1 \leq \max_j |q_j| \leq Q} ||q_1\alpha_1 + ... + q_n\alpha_n|| \leq \frac{1}{Q^n}.$$

b. Let $1, \alpha_1, ..., \alpha_n$ be linearly independent over $\mathbb{Z}$, that is

$$q_0 + q_1\alpha_1 + ... + q_n\alpha_n \neq 0 \quad \forall (q_0, q_1, ..., q_n) \in \mathbb{Z}^{n+1} \setminus \{(0, 0, ..., 0)\}.$$

Prove that there exist infinitely many vectors $(q_1, ..., q_n) \in \mathbb{Z}^{n+1} \setminus \{(0, 0, ..., 0)\}$ such that

$$||q_1\alpha_1 + ... + q_n\alpha_n|| \leq \frac{1}{(\max_{1 \leq j \leq n} |q_j|)^n}.$$

4. Golden section. Prove that for any $\varepsilon > 0$ the inequality

$$\left|\frac{\sqrt{5}+1}{2} - \frac{p}{q}\right| \leq \frac{1-\varepsilon}{\sqrt{5}\, q^2}.$$

has only finite number of solutions in fractions $\frac{p}{q} \in \mathbb{Q}$. (Suggestion: $q^2 \left|\frac{\sqrt{5}+1}{2} - \frac{p}{q}\right| \cdot \left|\frac{\sqrt{5}+1}{2} - \frac{p}{q} - \sqrt{5}\right| \in \mathbb{Z}_+$.)

5. Minimal polynomial. What are the degrees and the minimal polynomials for
   a) $\sqrt[3]{2}$?
   b) $\sqrt{2} + \sqrt{3}$?
(Suggestion for a.: $x^3 - 2$ has no rational roots.)

6. Degree of algebraic number. Prove that for any $n \in \mathbb{Z}_+$ there exists an algebraic α with $\deg \alpha = n$. (Suggestion: it is sufficient to find an irreducible over $\mathbb{Q}$ polynomial of degree n.)

7. Transcendental numbers. Prove that the numbers are not algebraic:

$$a. \sum_{n=0}^{\infty} \frac{1}{2^{2^{n^2}}}; \quad b. \sum_{n=0}^{\infty} \frac{1}{3^{n!}}.$$

8. Number e.
a. Prove $e \notin \mathbb{Q}$.
b. Prove that e is not a quadratic irrationality, that is $ae^2 + be + c \neq 0$ for every $(a, b, c) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}$. (Suggestion: $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$, and it is enough to prove that $ae+b+c/e \neq 0 \,\forall\, (a, b, c)$.)