## 7. Minkowski Question Mark function (24 July 2019).

For the Stern-Brocot sequences $F_n$ we consider the function

$$?(x) = \lim_{n\to\infty} \frac{\#\{\xi \in F_n : \ \xi \le x\}}{\#F_n}, \quad x \in [0,1], \tag{1}$$

which is called the Minkowski Question Mark function. By the way, a-pripori it is not clear why this limit exists. So the next statements 1, 2, 3, 4 ensure that it is really so.

1. For elements $\xi_{j,n}$ of the sequence $F_n$ one has $?(\xi_{j,n}) = \frac{j}{2^n}$.

2. The function $?(x)$ increases monotonically for rational values of $x$.

3. For irrational $x$ one has
$$\sup_{\mathbb{Q}\ni\frac{p}{q}<x} ?\left(\frac{p}{q}\right) \le \inf_{\mathbb{Q}\ni\frac{p}{q}>x} ?\left(\frac{p}{q}\right).$$

4. The limit in (1) exists.

5. $?(x)$ is continuous in every point.

6. **Lebesgue Theorem.** *Every monotone function on an interval $I$ has derivative in almost every (in the sense of Lebesgue measure) point of $I$.*
We will not prove this theorem. A. Lebesgue thought this theorem to be the best of his results. In particular, from this theorem it follows that $?(x)$ has derivative $?'(x)$ almost everywhere.

7. **Theorem 1.** *If at point $x \in [0,1]$ the derivative $?'(x)$ exists then either $?'(x) = 0$ or $?'(x) = +\infty$.*

8. What is $?'(x)$ for $x \in \mathbb{Q}$?

9. Stable points. The equation $?(x) = x$ has at least 5 solutions.

10. **Theorem 2.** *Let all the partial quotients in the continued fraction expansion of irrational $x$ are $\le 4$. Then $?'(x) = +\infty$.*

11. **Koksma's inequality.**
$$\left| \int_0^1 f(x)dx - \frac{1}{q}\sum_{j=1}^{q} f(\xi_j) \right| \le \frac{V[f]D(\Xi)}{q},$$
where $D(\Xi)$ is the discrepancy of the sequence $\Xi = (\xi_1, ..., \xi_q)$ and
$$V[f] = \sup_{x_1<x_2<...<x_t} \sum_{j=0}^{t-1} |f(x_{j+1}) - f(x_j)|$$
is the full variation of $f$.

12. **Theorem 3.**
$$\left| 2^n \cdot \int_0^1 ((?(x) - x)^2 dx - \sum_{j=0}^{2^n} \left(\xi_{j,n} - \frac{j}{2^n}\right)^2 \right| \le 4.$$

**Exercises.**

1. Prove that if $\frac{a}{b}, \frac{c}{d} \in [0,1]$ and $ad - bc = 1$ then

$$?\left(\frac{a}{b} \oplus \frac{c}{d}\right) = \frac{1}{2}\left(?\left(\frac{a}{b}\right) + ?\left(\frac{c}{d}\right)\right).$$

2. a. For irrational $x = [0; a_1, a_2, ..., a_n, ...]$ prove the formula

$$?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + ... + \frac{(-1)^{n+1}}{2^{a_1+...+a_n-1}} + ....$$

b. What is he analogous formula for *rational* $x = [0; a_1, a_2, ..., a_n]$?

3. Prove that if $x$ is a quadratic irrationality then $?(x) \in \mathbb{Q}$.

4. Prove Theorem 2.
Three simplifications of this problem.
a. Prove that $?'\left(\frac{\sqrt{5}-1}{2}\right) = +\infty$.
b. Prove that $?'(x) = +\infty$ for all irrational $x$ with partial quotients $\leq 2$.
c. Prove that $?'(x) = +\infty$ for all irrational $x$ with partial quotients $\leq 3$.

5. Find $?'(x_d)$ for every $x_d = [0; \overline{d}]$.

6. Prove that the equation $?(x) = x$ an irrational solution.

7. How many solutions has equation $?(x) = x$? (Unsolved and probably hopeless.)