## 8. Geometry of Continued Fractions (26 July 2019).

We suppose for simplicity that $\alpha > 0$ and $\alpha \notin \mathbb{Q}$.

1. Klein polygons. In the first quadrant $\mathbb{R}^2_{++} = \{x \geq 0, y \geq 0\}$ we consider the line

$$y = \alpha x$$

and the angles

$$\varphi_- = \{0 \leq y \leq \alpha x\}, \quad \varphi_+ = \{0 \leq x \leq y/\alpha\}.$$

Consider the *convex hulls*

$$\operatorname{conv}\left(\varphi_\pm \cap \mathbb{Z}^2\right)$$

and *non-trivial* parts of the boundaries

$$\mathcal{K}_\pm = \partial \operatorname{conv}\left(\varphi_\pm \cap \mathbb{Z}^2\right).$$

Infinite broken lines $\mathcal{K}_\pm$ are called Klein polygons.

2. *Separating line* for a set $\Omega \subset \mathbb{R}^2$ is a line $\ell$ such that $\ell \cap \Omega \neq \varnothing$ and $\Omega$ completely belongs to one of half-planes with boundary $\ell$. If $\Omega$ is a polygon and there exists more than one separating line passing through $x \in \Omega$, then $x$ is a vertex of $\Omega$.

3. **Theorem.** *Vertices of $\mathcal{K}_\pm$ are of the form $(q_\nu, p_\nu)$ and there are no other vertices.*

4. Empty parallelogram. Consider parallelogram with integer vertices

$$(0,0), \quad (a,b), \quad (c,d), \quad (a+c, b+d)$$

and area 1. Let the line $\{y = \alpha x\}$ divides this parallelogram into two parts. One of non-zero vertices belongs to one part and two other to another part. Then

a) all the vertices are on the boundary of $\mathcal{K}_\pm$;

b) the lonely vertex is a vertex of $\mathcal{K}_\pm$.

Compare with the following fact we discussed: if $\alpha \in \left[\frac{a}{b}, \frac{c}{d}\right]$ and the mediant fraction $\frac{a+c}{b+d}$ falls into $\left[\alpha, \frac{c}{d}\right]$, then $\frac{a}{b}$ is a convergent to $\alpha$.

5. Best approximations. A vector $(q,p) \in \mathbb{Z}^2$ is called *best approximation* (B.A) if

$$|q\alpha - p| = \min_{(q',p') \in \mathbb{Z}^2 \setminus \{(0,0)\} : 1 \leq q' \leq q} |q'\alpha - p'|.$$

**Theorem.** *A vector $(q,p) \in \mathbb{Z}^2$ is B.A. if and only if it is a vertex of $\mathcal{K}_\pm$ and hence a continued fraction convergent to $\alpha$.*

6. Legendre theorem again. Prove that if $(p,q) = 1$ and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2} \tag{1}$$

then $\frac{p}{q}$ is a convergent to $\alpha$.

7. **Fatou Theorem.** *If $(p,q) = 1$ and $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$, then either $\frac{p}{q}$ is a convergent to $\alpha$ or is of the form $\frac{p}{q} = \frac{p_n \pm p_{n-1}}{q_n \pm q_{n-1}}$ for some $n$.*

7. **Minkowski Theorem.** *For a point* $\boldsymbol{z} = (q, p)$ *consider the point*

$$\boldsymbol{z}^* = (q, p + 2(\alpha q - p)).$$

*Let* $\boldsymbol{z}$ *be a vertex of* $\mathcal{K}_\pm$. *Then (1) holds if and only if* $\boldsymbol{z}^*$ *is strictly between the Klein polygon and the line* $y = \alpha x$.