10. Hall's Ray (30 July 2019 – 01 August 2019).

Theorem. $\exists\,\lambda^* > 0 : \quad [0, \lambda^*] \subset \mathbb{L}.$

This theorem follows from the

Main Lemma. *Suppose that*

$$K_r = \{\alpha \in [0,1] \setminus \mathbb{Q};\ \alpha = [0; a_1, ..., a_n, ...] : \quad a_j \leq r\ \forall j\}.$$

Then $K_4 + K_4$ *is a segment.*

Plan of the proof of Main Lemma.

A. What is the definition of τ-thick set

$$K = I \setminus \left(\bigcup_j \Delta_j \right),$$

where I is a segment and $\Delta_j \subset I$ are subintervals of I which do not intersect each other?
B. If K is a 1-thick set, than $K + K$ is a segment.
C. K_4 is an 1-thick set.

Exercises.

1. Prove that if

$$\alpha = [0; a_1, ..., a_n, x], \quad \beta = [0; a_1, ..., a_n, y]$$

then

$$|\alpha - \beta| = \frac{|x - y|}{(xq_n + q_{n-1})(yq_n + q_{n-1})}.$$

2. a. Prove that there exists an irrational number which can be written in the base 3 without digit 1 and simultaneously in the base 5 without digit 3.

b. Prove that there exist a number α of the form $\alpha = [0; a_1, ..., a_n, ...] : \quad a_j \leq 4$, which can be written in the base 3 without digit 1.

3. Suppose that $\tau_1 \cdot \tau_2 \geq 1$. Prove that any two τ_1-thick set and τ_2-thick set have non-empty intersection provided that one does not belong to a connected part of the complement to another one.

4. How thick is a. K_2? b. K_3?

5. Prove that $K_3 + K_3 + K_3$ is a segment.

6. How many copies of K_2 one should take to have the sum $K_2 + ... + K_2$ to be a segment?

7. Prove that if a set is a τ-thick set we can throw away intervals Δ_j from I in such a way, that their lengths are decreasing. (This will preserve τ-thickness property.)