LINEAR ALGEBRA HOMEWORK 2 - DUE FRIDAY 7/26

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While you should always try all assigned problems, you should write up only the ones you are asked to write up.

Exercise 0.1. *Assume we have a field* $\mathbb{C}$ *with properties* ($\mathbb{C}1$)*-*($\mathbb{C}3$)*. Use N1-N9 to verify that for* $z, w \in \mathbb{C}$, $z + w = 0$ *iff* $w = -z$*. Now show that for* $x, y \in \mathbb{R}$

$$-(x + iy) = -x + i(-y).$$

In fact, in any field F*, N1-N9 imply that* $-(a + b) = (-a) + (-b)$ *and* $-(ab) = (-a)b$*.*

Assume U, V, W are F-vector spaces.

Exercise 0.2. *As in class, let* $\mathrm{Hom}(U, V) \equiv \mathrm{Hom}_F(U, V)$ *be the set of* F*-linear maps* $U \to V$*. Consider the MMC map*

$$\Phi : M_{k,l} \to \mathrm{Hom}(F^l, F^k), \quad A \mapsto L_A$$

Treat tbe matrix space $M_{k,l}$ *as a* F*-vector space as usual. Show that there is just one way to make* $\mathrm{Hom}(F^l, F^k)$ *a* F*-space, such that* Φ *is linear. More generally, how would you make* $\mathrm{Hom}(U, V)$ *a* F*-vector space?*

Exercise 0.3. *If* $f : U \to V$ *and* $g : V \to W$ *are linear maps, verify that their composition* $gf \equiv g \circ f : U \to W$ *is also linear.*

Exercise 0.4. *Let* $\mathrm{Iso}(V, V)$ *be the set of isomorphisms (i.e. linear bijections)* $\phi : V \to V$*, and* $\mathrm{Iso}(V, W)$ *the set of isomorphisms* $f : V \to W$*. Suppose* $f_0 \in \mathrm{Iso}(V, W)$*. Show that the map*

$$T : \mathrm{Iso}(V, V) \to \mathrm{Iso}(V, W), \ \phi \mapsto f_0 \circ \phi$$

bijects, by writing the inverse map.

Exercise 0.5. *Describe sol*(E) *to the following system* E *in* $\mathbb{R}^4$*, using row reduction and then giving an isomorphism* $f : \mathbb{R}^\ell \to \ker L_A$ *(including specifying the appropriate* ℓ*), where* A *is the coefficient matrix* *of the system:*

$$E_0 : \quad \begin{aligned} x+y+z+t&=0 \\ x+y+2z+2t&=0 \\ x+y+2z-t&=0. \end{aligned}$$

Replace the 0 *on the right side of first equation by* 1 *to get a new inhomogeneous system* E_1*, and then describe its solution set* $sol(E_1)$ *by writing down an explicit translation map.*

Exercise 0.6. *WRITE UP Let* E *be an* n*-variable* F*-linear system. Prove that*

$$\begin{aligned} & E \text{ is homogenous} \\ \Leftrightarrow\ & 0 \in sol(E) \\ \Leftrightarrow\ & sol(E) \text{ is } F\text{-subspace of } F^n. \end{aligned}$$

Try to make your proof as simple as possible, say less than half a page.

Exercise 0.7. *WRITE UP Let* $F[x]_d$ *be the* F*-subspace of* $F[x]$ *consisting of all polynomials* $p(x)$ *of degree at most* d*, i.e. the highest power* x^n *appearing in* $p(x)$ *is at most* x^d*. Consider the map*

$$L_{n,d} := (1-x^2)(\frac{d}{dx})^2 - 2x\frac{d}{dx} + n(n+1)id : F[x]_d \to F[x]_d$$

for integer $n \geq 0$*. Verify that* $L_{n,d}$ *is* F*-linear. Describe* $\ker L_{n,d}$ *by solving the linear equation*

$$L_{n,d}(f) = 0$$

for $n = 0, 1, 2$*, say by giving a basis of* $\ker L_{n,d}$*. Equivalently, find* $k \in \mathbb{Z}_{\geq 0}$ *(which can depend on* n, d*) such that you can construct an* F*-isomorphism*

$$f : F^k \to \ker L_{n,d}.$$