

# LINEAR ALGEBRA HOMEWORK 3 – DUE 7/29

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$F$  denotes a field. Assume  $U, V, W$  are  $F$ -vector spaces, and all dimensions are  $F$ -dimensions.

**Exercise 0.1.** Let  $V$  be a  $F$ -subspace of  $F^n$ . Decide whether each of the following is TRUE of FALSE. Justify your answer. For (a)-(e), assume that  $\dim V = 3$ .

- (a) Any 4-tuple of  $V$  is linearly dependent.
- (b) Any 2-tuple of  $V$  is linearly independent.
- (c) Any 3-tuple of  $V$  is a basis.
- (d) Some 3-tuple of  $V$  is a basis.
- (e)  $V$  contains a linear subspace  $W$  with  $\dim W = 2$ .
- (f)  $(1, \pi), (\pi, 1)$  form a basis of  $\mathbb{R}^2$ . You can assume that  $|\pi - 3.14| < 0.01$ .
- (g)  $(1, 0, 0), (0, 1, 0)$  do not form a basis of the plane  $x - y - z = 0$ .
- (h)  $(1, 1, 0), (1, 0, 1)$  form a basis of the plane  $x - y - z = 0$ .
- (i) If  $A$  is a  $3 \times 4$  matrix, then the subspace  $V$  of  $F^4$  generated by the rows of  $A$  is at most 3 dimensional.
- (j) If  $A$  is a  $4 \times 3$  matrix, then the subspace  $V$  of  $F^3$  generated by the rows of  $A$  is at most 3 dimensional.

**Exercise 0.2.** WRITE UP Let

$$V = \{(a + b, a, c, b + c) | a, b, c \in F\} \subset F^4.$$

Verify that  $V$  is an  $F$ -subspace of  $F^4$ . Find a basis of  $V$ .

**Exercise 0.3.** WRITE UP Do this without using COD.

- (a) In 2 lines, show that any 3-tuple in  $V = F^2$  is dependent. In 3 line, generalize this to any  $F$ -space  $V$  with  $\dim V = n < +\infty$ .
- (b) Let  $U \subset V$  be an  $F$ -subspace, and  $\mathcal{S}$  the set of all independent tuples in  $U$ . In 5 lines, show that if  $\dim V = n < +\infty$  then  $\mathcal{S}$  contains a basis of  $U$ , and conclude that  $\dim U \leq \dim V$ .  
(Hint: Consider a tuple of **maximal** length in  $\mathcal{S}$ ; why does it exists?)
- (c) In 5 lines show that if furthermore  $\dim U = \dim V$  then  $U = V$ .

**Exercise 0.4.** Fix  $0 < k < n$  and consider the decomposition

$$F^n \cong F^k \oplus F^{n-k}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cong \begin{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \\ \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} u_k \\ \ell_{n-k} \end{bmatrix}.$$

Show if  $A \in M_{n,n}$ , then  $A$  also ‘decomposes’ into a corresponding a block form

$$A \equiv \begin{bmatrix} P_{k,k} & Q_{k,n-k} \\ R_{n-k,k} & S_{n-k,n-k} \end{bmatrix}$$

so that the column vector  $Ax$  can be expressed in terms of the column vectors  $Pu, Q\ell, Ru, S\ell$ . If you are confused, do the special case  $n = 3, k = 1$  first.

**Exercise 0.5.** *WRITE UP* In 1 line, prove that every matrix  $A \in M_{n,n}$  satisfies a nontrivial polynomial equation of the form

$$a_0 I_n + a_1 A + \cdots + a_k A^k = 0.$$

**Exercise 0.6.** Find a basis of  $\text{sol}(E)$  in  $F^4$  for

$$E : x - y + 2z + t = 0.$$

**Exercise 0.7.** Find a basis for each of the subspaces  $\ker L_A$  and  $\text{im } L_A$  of  $F^4$ , where  $A$  is the matrix

$$\begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & -2 & 4 & 2 \\ 1 & 0 & 1 & 1 \\ 3 & 4 & -5 & -1 \end{bmatrix}.$$

**Exercise 0.8.** We know that  $V^2 = V \times V$  form a vector space. Define an  $F$ -vector space structure on  $U \times V$  a vector space. Let’s call it the **direct sum**  $U \oplus V$  of  $U, V$ . If  $\dim U = k$  and  $\dim V = n$ , what is  $\dim(U \oplus V)$ ? Prove your assertion in 5 lines.

**Exercise 0.9.** (Revisit MMC) We specialize to the case  $V = F^2$ . Let  $(v_1, v_2) \in V^2$ , put  $A = [v_1, v_2] \in M_{2,2}$ , and write  $v_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$ ,  $v_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$ .

(a) (A numerical test for isomorphism) Show that  $v_1, v_2$  are ‘parallel’, i.e. one a scalar multiple of the other, iff they are dependent, iff

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

iff  $L_A$  is not an isomorphism, iff  $(v_1, v_2)$  is not a basis of  $V$ .

(b) Now suppose  $L_A$  is an isomorphism. Can you find the matrix  $B$  corresponding to (under MMC) to the inverse isomorphism  $L_A^{-1} : F^2 \rightarrow F^2$ ?