Example 1.3.11 (Erdős)

Every n points in the plane which are not all collinear must determine at least n distinct lines.

§1.4 Additional Reading

See Sections 1.1-1.4 and 3.1-3.3 from the textbook. For sake of clarity, this is the reference [MN] which was also cited above, namely "Invitation to Discrete Mathematics" (2nd ed.) by Jiri Matousek and Jaroslav Nesetril.

§1.5 Problems

These are some problems to practice the material above and do **not** represent homework unless explicitly mentioned otherwise. Give them a try! Some of them will be discussed by your TA during the upcoming discussion sessions from 4 to 5 PM on Tuesdays and Thursdays (starting with September 8th).

Problem 1.5.1. How many subsets of $\{1, \ldots, n\}$ are there which do not contain the element 5?

Problem 1.5.2. How many permutations of $\{1, \ldots, n\}$ are cyclic, in the sense of Definition 1.1.7?

Problem 1.5.3. How many permutations of $\{1, \ldots, n\}$ have the property that they fix precisely k elements? In other words, how many bijective functions $f : [n] \to [n]$ are there with the property that $f(x) = x$ for precisely k values of $x \in [n]$. Here $[n]$ is convenient notation for $\{1, \ldots, n\}$.

Problem 1.5.4. Let $m \geq n \geq 1$ be integers. How many strictly increasing functions $f : [n] \to [m]$ are there?

Problem 1.5.5. Prove that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ for all positive integers $n \geq k$.

Problem 1.5.6. Show that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$ for all positive integer $n \geq k \geq m$.

Problem 1.5.7. Prove that:

$$\sum_{k \geq 0} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}.$$

Can you also find a combinatorial argument by counting a quantity in two different ways directly?

Problem 1.5.8. Let n be a positive integer. Show that

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Problem 1.5.9. Compute $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$.

Problem 1.5.10. Compute $\sum\limits_{k=0}^{n} k^2\binom{n}{k}$.

Problem 1.5.11. Prove the formula

$$\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\ldots+\binom{n}{r}=\binom{n+1}{r+1}$$

by induction on n (for r arbitrary but fixed). Note that what the formula says for $r = 1$. Can you also prove the same formula combinatorially?

Problem 1.5.12. Use the previous problem to calculate the sums $\sum_{i=1}^{n} i^2$ and $\sum_{i=1}^{n} i^3$ in terms of n.

Problem 1.5.13. Consider a pentagon in the plane whose vertices have integer cartesian coordinates. Prove that the midpoint of one of its diagonals must also have integer coordinates. More generally, prove that every set of $2^n + 1$ vectors in $\mathbb{Z}^n$ (integer coordinates) contains a pair of distinct points whose midpoint also has integer coordinates.

Problem 1.5.14. Consider five poins in a square with side length 2. Prove that no matter how these points are placed, some pair of them are no more than $\sqrt{2}$ apart.

Problem 1.5.15. A chess grandmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more then 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.

Problem 1.5.16. Prove that every polygon can be dissected into triangles by interior diagonals.

Problem 1.5.17. Prove Bernoulli's inequality: if $x \geq -1$ is a real number, then $(1+x)^n \geq 1 + nx$ for all natural numbers n.

Problem 1.5.18. Recall the so-called *harmonic numbers* from Math 115:

$$H_k = 1 + \frac{1}{2} + \ldots + \frac{1}{k}$$

for $k \geq 1$. Use induction to prove that

$$H_{2^n} \geq 1 + \frac{n}{2}$$

for every $n \geq 1$.

Problem 1.5.19. Given is a list of n positive integers whose sum is less than $2n$. Prove that, for any positive integer m not exceeding the sum of these integers, one can choose a sublist of the integers whose sum is m.