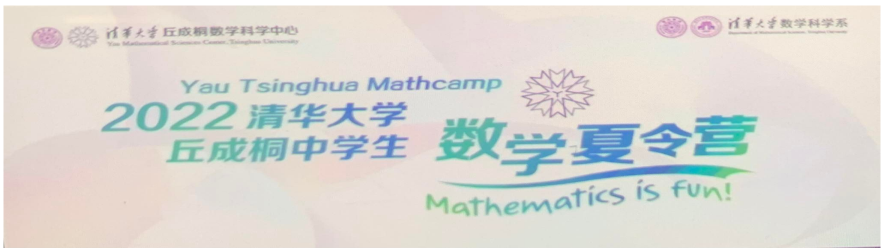

ALGEBRAIC COMBINATORICS I, YAU TSINGHUA MATHCAMP, SUMMER 2022

RESEARCH PROJECTS

1. Sperner's lemma and Brouwer's fixed point theorem

In 1928, Emanuel Sperner found a surprisingly simple proof of Brouwer's fixed point theorem whcih states that every continuous map of an n-dimensional ball to itself has a fixed point. At the heart of his proof is the following combinatorial lemma. First, we need to define the notions of simplicial subdivision and proper coloring.

Definition 1.1. *An n-dimensional simplex is a convex linear combination of $n+1$ points in a general position, i.e. for given vertices $v_1, v_2, \ldots, v_{n+1}$, the simplex would be*

$$S = \left\{ \sum_{i=1}^{n+1} \alpha_i v_i : \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

A simplicial subdivision of an n-dimensional simplex S is a partition of S into small simplices (cells) siuch that any two cells are either disjoint, or they share a full face of certain dimension.

Definition 1.2. *A proper coloring of a simplicial subdivision is an assignment of $n+1$ colors to the vertices of the subdivision, so that the vertices of S receive all different colors, and points on each face of S use only the colors of the vertices defining the respective face of S.*

For example, for $n = 2$ we have a subdivision of a triangle T into triangular cells. A proper coloring of T assigns different colors to the 3 vertices of T, and inside vertices on each edge of T use only the two colors of the respective endpoints.

Lemma 1.3 (Sperner, 1928). *Every properly colored simplicial subdivision contains a cell whose vertices have all different colors.*

You are asked to **prove Lemma 1.3**. As a hint, try to treat the cases $n = 1$ and $n = 2$ separately. A groundbreaking application of Sperner's lemma is given by

Theorem 1.4 (Brouwer, 1911). *Let B^n be the n-dimensional ball. Show that for any continuous map $f : B^n \to B^n$, there is a fixed point, namely $x \in B^n$ such that $f(x) = x$.*

You are asked to **prove Theorem 1.4**. As a hint, try to use of Sperner's lemma (Lemma 1.3) as a key ingredient.

2. The Borsuk-Ulam antipodal theorem, Kneser graphs and Lovasz-Kneser theorem

A more powerful topological tool than Brouwer's theorem is the so-called Borsuk-Ulam theorem which is a very important tool in combinatorics and algebraic topology. Let us denote S^n be the n-dimensional sphere (the surface of the $(n+1)$-dimensional ball).

Theorem 2.1 (Borsuk-Ulam). *For any continuous map $f : S^n \to \mathbb{R}^n$, there is a point $x \in S^n$ such that $f(x) = f(-x)$.*

The proof of this theorem uses algebraic topology. In what follows we will give a restatement of Theorem 2.1.

Theorem 2.2. *For any covering of S^n by $n+1$ open or closed sets $A_0, \dots, A_n$, there is a set A_i which contains two antipodal points, x and $-x$.*

You are asked to **show how Theorem 2.2 follows from Theorem 2.1**.

Kneser graphs are derived from intersection pattern of a collections of sets. More precisely, the vertex of Kneser graph consists of all k-sets on a given ground set, and two k-sets form an edge if they are disjoint.

Definition 2.3. *The Kneser graph on a ground set $[n]$ is*

$$KG_{n,k} = \left(\binom{[n]}{k}, \{(A, B) : |A| = |B| = k, A \cap B = \emptyset\} \right)$$

Thus, the maximum independent set in $KG_{n,k}$ is equivalent to the maximum intersecting family of k-sets by Erdos-Ko-Rado theorem, we have $\alpha(KG_{n,k}) = \binom{n-1}{k-1} = \frac{k}{n}|V|$ for $k \le \frac{n}{2}$. The maximum clique in $KG_{n,k}$ is equivalent if disjoint k-sets, i.e. $\omega(KG_{n,k}) = [n/k]$.

Another natural questions is, what is the chromatic number of $KG_{n,k}$? Note that for $n = 3k - 1$, the Kneser graph does not have any triangle.

Theorem 2.4 (Lovasz-Kneser). *For all $k > 0$ and $n \ge 2k - 1$, we have $\chi(KG_{n,k}) = n - 2k + 2$.*

Prove Theorem 2.4. As a hint, try to use Theorem 2.2.

3. Generating functions and Euler's Basel problem

The Basel problem is one of the most famous problems in analysis and number theory and it concerns the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

The result was proved by Euler in 1734 and is given by

Theorem 3.1 (Euler)**.** *We have*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this project you will be guided to obtain a proof of this result via generating functions.

First, **prove** that the function $y(x) = \arcsin^2 x$ verifies the second order initial value problem,

$$(1-x^2)y'' - xy' - 2 = 0, y(0) = y'(0) = 0.$$

Second, when we look for the power series solution of the above initial value problem, $y(x) = \sum_{n\geq 0} a_n x^n$, **prove** that

$$\arcsin^2 x = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2\binom{2n}{n}}, |x| \leq 1.$$

Third, with a little bit of extra care, **prove** the convergence of the series above.

Fourth, **prove** the following Wallis' formula,

$$\int_0^{\frac{\pi}{2}} \sin^{2n} t dt = \frac{\pi}{2^{2n+1}}\binom{2n}{n}.$$

Last but not least, combine all these previous steps, and **derive** the proof of the Basel problem.

4. Applications of probabilistic method in Combinatorics: The inequalities of Bollobas, Tusza and Alon

Solve Problems 6.7.10, 6.7.11 and 6.7.12 from Homework no. 6. These are important results of the probabilistic method in combinatorics.

5. The friendship theorem

The friendship theorem can be interpreted in a popular language as follows. If any two people have exactly one friend in common, then there is a person who is everybody's friend.

Surprisingly, the friendship theorem is false for infinite graphs (**prove it!**). Although it is somewhat similar to Erdos-Ko-Rado theorem, the result below requires some spectral analysis.

Theorem 5.1 (Friendship theorem)**.** *Suppose G is a finite graph where any two vertices share exactly one neighbor. Then there is a vertex adjacent to all other vertices.*

You are asked to **prove Theorem 5.1**.

6. Chebyshev's inequality and vectors in the plane

Chebyshev's inequality is a very important tool in probability theory as well as in the probabilistic methods in combinatorics. The statement is as follows.

Theorem 6.1 (Chebyshev's inequality)**.** *Let X be a random variable and let $a > 0$. Then*

$$P(|X - E(X)| \geq a) \leq \frac{1}{a^2}(E(X^2) - E(X)^2).$$

This theorem will be proved in class in the last chapter. As an application, you are asked to **provide a solution for Problem 6.7.13 from Homework no. 6**.

7. Markov's inequality, graphs of high girth and high chromatic number

Another simple tool from probability theory which bounds the probability that a random variable X is too large, based on the expectation $E(X)$.

Theorem 7.1 (Markov's inequality)**.** *Let X be a nonnegative random variable and $a > 0$. Then*

$$P(X \geq a) \leq \frac{1}{a}E(X).$$

This result will be proved in last week's lectures. Although it is a result in probability theorem, it seems that it has applications in graph theory regarding the chromatic number, $\chi(G)$.

Definition 7.2. *For a graph G, the chromatic number $\chi(G)$ is the smallest c such that the vertices of G can be colored with c colors with no neighboring vertices receiving the same color.*

We know that for a graph that does not contain any cycles, we have $\chi(G) \leq 2$. This is true because every component is a tree that can be colored easily by 2 colors.

More generally, if we consider a graph of girth l, i.e. the length of the shortest cycle is l. Now, if l is large, this means that starting from any vertex, the graph would look like a tree within distance $l/2 - 1$. One might expect that such graphs can be also colored using s small number of colors, since locally they can be colored using 2 colors. The next result shows that this is far for being true.

Theorem 7.3. *For any k and l, there is a graph of chromatic number and girth strictly greater than k.*

Prove Theorem 7.3.

8. Cauchy-Davenport theorem and Erdos-Ginzburg-Ziv theorem

A nice application of the principle of inclusion-exclusion is given by the following

Theorem 8.1 (Cauchy-Davenport)**.** *Let $p \geq 3$ be a prime number. Then for any two nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$ we have*

$$|A+B| \geq \min(|A|+|B|-1, p).$$

Here $A+B = \{a+b : a \in A, b \in B\}$.

You are asked to **prove Theorem 8.1**. Moreover, this theorem can be generalized as follows.

For $p \geq 3$ prime, and $A_1, A_2, \ldots, A_k$ subsets of $\mathbb{Z}/p\mathbb{Z}$ we have

$$|A_1 + A_2 + \ldots A_k| \geq \min(|A_1|, |A_2|, \ldots, |A_k| - k + 1, p).$$

Using this generalization, you are asked to **prove** the following

Theorem 8.2 (Erdos-Ginzburg-Ziv)**.** *For any $2n-1$ integers we can choose n of them such that their arithmetic mean is an integer.*

9. Extremal combinatorics: The Erdos-Stone theorem

A natural question to ask is what is the maximum number of edges in a graph G on n vertices, which does not contain a given subgraph H?

We denote this by $ex(n, H)$. For graphs G on n vertices, this question is resolved up to an additive error $o(n^2)$ by the so-called Erdos-Stone theorem.

First, let us recall the definition of a chromatic number.

Definition 9.1. *For a graph H, the chromatic number $\chi(H)$ is the smallest c such that the vertices of G can be colored with c colors with no neighboring vertices receiving the same color.*

The chromatic number is a very important parameter of a graph. The graphs of chromatic number at most 2 are exactly bipartite graphs. In contrast, graphs of chromatic number 3 are already hard to describe and hard to recognize algorithmically. For example, the famous four color theorem which states that any graph that can be drawn in the plane without crossing edges has chromatic number at most 4.

The chromatic number turns out to be closely related to the question of how many edges are necessary for H to appear as a subgraph.

Theorem 9.2 (Erdos-Stone)**.** *For any fixed graph H and fixed $\epsilon > 0$, there is n_0 such that for any $n \geq n_0$ we have*

$$\frac{1}{2}\left(1 - \frac{1}{\chi(H)-1} - \epsilon\right) n^2 \leq ex(n, H) \leq \frac{1}{2}\left(1 - \frac{1}{\chi(H)-1} + \epsilon\right) n^2$$

In particular, for bipartite graphs H, which can be colored with 2 colors, we get that $ex(H,n) \leq \epsilon n^2$ for any $\epsilon > 0$ and sufficiently large n, so Theorem 10.2 only says that the extremal number is very small compared to n^2. We denote this by $ex(H,n) = o(n^2)$. For graphs H of chromatic number 3, we get $ex(n,H) = \frac{1}{4}n^2 + o(n^2)$.

First, you are asked to **prove the following Lemma 9.3** given below

Lemma 9.3. *Fix $k \geq 1$, $0 < \epsilon < 1/k$ and $t \geq 1$. Then there is $n_0(k,\epsilon,t)$ such that any graph G with $n \geq n_0(k,\epsilon,t)$ vertices and $m \geq \frac{1}{2}(1 - 1/k + \epsilon)n^2$ edges contains $k+1$ disjoint sets of vertices $A_1, A_2, \ldots, A_{k+1}$ of size t, such that any two vertices in different sets A_i, A_j are joined by an edge.*

You are asked to **prove Theorem 9.2**. You can use Lemma 9.3 as a key ingredient.

10. Variational definition of eigenvalues and the independence number

Linear algebra is an essential tool in dealing with combinatorial problems. Apart from the classical definition of an eigenvalue of a a matrix, we give an equivalent variational characterization of eigenvalues.

Lemma 10.1. *The k-th largest eigenvalue of a matrix $A \in M_n(\mathbb{R})$ is given by*

$$\lambda_k = \max_{\dim(U)=k} \min_{x \in U} \frac{x^T A x}{x^T x} = \min_{\dim(U)=k-1} \max_{x \perp U} \frac{x^T A x}{x^T x}.$$

Here the maximum/minimum is taken over all subspaces U of a given dimension, and over all nonzero vectors x in the respective subspace.

You are asked to **prove Lemma 11.1**. Using the above result (Lemma 10.1) one can find an upper bound for the indepedence number.

Theorem 10.2. *For a d-regular graph with smallest (most negative) eigenvalue λ_n, the independence number is*

$$\alpha(G) \leq \frac{n}{1 - d/\lambda_n}.$$

You are asked to **prove Theorem 10.2**.