LINEAR ALGEBRA HOMEWORK 2

LECTURER: BONG H. LIAN

While you should always try all assigned problems, you should write up only the ones you are asked to write up.

Exercise 0.1. WRITE UP *Let F be a field. Recall that E a subfield of F if E is subset of F that is closed under the 4 algebraic operations of F $(+,-,\times,1/)$ and contains $0,1$ of F. Now suppose E is a subfield of F, and S is a subset of F. In 10 lines, prove that there exists a unique subfield $E(S)$ with the following properties: $E(S)$ contains S; if K is any subfield of F containing E and S, then K contains $E(S)$. In other words $E(S)$ is the 'smallest' subfield F containing E and S.*

Exercise 0.2. *Show that if F is a field of characteristic 0, then F contains the field $\mathbb{Q}$ as a subfield in some sense. Make precise in what sense this is true. Similarly, if F is a field of characteristic prime p, then F contains the field $\mathbb{F}_p$ of p elements as a subfield.*

Exercise 0.3. WRITE UP *Consider the field $F = \mathbb{R}$, and $\alpha \in \mathbb{R}$ a root of a polynomial with coefficients in $\mathbb{Q}$, say*

$$p(t) = a_0 + a_1 t + \cdots + a_d t^d$$

where $a_0, .., a_d \in \mathbb{Q}$ and $a_d \neq 0$. Let $S = \{\alpha\}$. Can you describe the field $\mathbb{Q}(S)$ in terms of α?
Idea: Does $\mathbb{Q}(S)$ contain $1, \alpha, \alpha^2, ...$?

Assume U, V, W are F-vector spaces.

Exercise 0.4. *If $f : U \to V$ and $g : V \to W$ are linear maps, verify that their composition $gf \equiv g \circ f : U \to W$ is also linear.*

Exercise 0.5. *Let $\mathrm{Iso}(V, V)$ be the set of isomorphisms (i.e. linear bijections) $\phi : V \to V$, and $\mathrm{Iso}(V, W)$ the set of isomorphisms $f : V \to W$. Suppose $f_0 \in \mathrm{Iso}(V, W)$. Show that the map*

$$T : \mathrm{Iso}(V, V) \to \mathrm{Iso}(V, W), \ \phi \mapsto f_0 \circ \phi$$

bijects, by writing the inverse map.

Exercise 0.6. *Describe $sol(E)$ to the following system E in $\mathbb{R}^4$, using row reduction and then giving an isomorphism $f : \mathbb{R}^\ell \to \ker L_A$ (including specifying the appropriate ℓ), where A is the coefficient matrix of the system:*

$$E_0 : \quad \begin{aligned} x+y+z+t&=0\\ x+y+2z+2t&=0\\ x+y+2z-t&=0. \end{aligned}$$

Replace the 0 on the right side of first equation by 1 to get a new inhomogeneous system E_1, and then describe its solution set $sol(E_1)$ by writing down an explicit translation map.

Exercise 0.7. *Let E be an n-variable F-linear system. Prove that*

$$\begin{aligned} &E \text{ is homogenous}\\ \Leftrightarrow\ & 0 \in sol(E)\\ \Leftrightarrow\ & sol(E) \text{ is } F\text{-subspace of } F^n. \end{aligned}$$

Try to make your proof as simple as possible, say less than half a page.

Exercise 0.8. *Let $F[x]_d$ be the F-subspace of $F[x]$ consisting of all polynomials $p(x)$ of degree at most d, i.e. the highest power x^n appearing in $p(x)$ is at most x^d. Consider the map*

$$L_{n,d} := (1-x^2)(\frac{d}{dx})^2 - 2x\frac{d}{dx} + n(n+1)id : F[x]_d \to F[x]_d$$

for integer $n \geq 0$. Verify that $L_{n,d}$ is F-linear. Describe $\ker L_{n,d}$ *by solving the linear equation*

$$L_{n,d}(f) = 0$$

for $n = 0, 1, 2$, say by giving a basis of $\ker L_{n,d}$. *Equivalently, find $k \in \mathbb{Z}_{\geq 0}$ (which can depend on n, d) such that you can construct an F-isomorphism*

$$f : F^k \to \ker L_{n,d}.$$