LINEAR ALGEBRA HOMEWORK 3

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F denotes a field. Assume U, V, W are F-vector spaces, and all dimensions are F-dimensions.

Exercise 0.1. *Let V be a F-subspace of F^n. Decide whether each of the following is TRUE of FALSE. Justify your answer. For (a)-(e), assume that* $\dim V = 3$.
(a) Any 4-tuple of V is linearly dependent.
(b) Any 2-tuple of V is linearly independent.
(c) Any 3-tuple of V is a basis.
(d) Some 3-tuple of V is a basis.
(e) V contains a linear subspace W with $\dim\ W = 2$.
(f) $(1, \pi)$, $(\pi, 1)$ form a basis of $\mathbb{R}^2$. You can assume that $|\pi - 3.14| < 0.01$.
(g) $(1, 0, 0)$, $(0, 1, 0)$ do not form a basis of the plane $x - y - z = 0$.
(h) $(1, 1, 0)$, $(1, 0, 1)$ form a basis of the plane $x - y - z = 0$.
(i) If A is a 3×4 matrix, then the subspace V of F^4 generated by the rows of A is at most 3 dimensional.
(j) If A is a 4×3 matrix, then the subspace V of F^3 generated by the rows of A is at most 3 dimensional.

Exercise 0.2. *WRITE UP Let*

$$V = \{(a + b, a, c, b + c)|a, b, c \in F\} \subset F^4.$$

Verify that V is an F-subspace of F^4. Find a basis of V.

Exercise 0.3. *Fix $0 < k < n$ and consider the decomposition*

$$F^n \equiv F^k \oplus F^{n-k}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \equiv \begin{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \\ \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \end{bmatrix} = \begin{bmatrix} u_k \\ \ell_{n-k} \end{bmatrix}.$$

Show if $A \in M_{n,n}$, then A also 'decomposes' into a corresponding block form

$$A \equiv \begin{bmatrix} P_{k,k} & Q_{k,n-k} \\ R_{n-k,k} & S_{n-k,n-k} \end{bmatrix}$$

so that the column vector Ax can be expressed in terms of the column vectors $Pu, Q\ell, Ru, S\ell$. If you are confused, do the special case $n = 3, k = 1$ first.

Exercise 0.4. *WRITE UP Let $A \in M_{n,n}(F)$. In 2 lines, prove that the F-algebra $F[A]$ of polynomials of A has dimension at most n^2. Conclude that A satisfies a nontrivial polynomial equation of the form*

$$a_0 I_n + a_1 A + \cdots + a_k A^k = 0.$$

Exercise 0.5. *Find a basis of* $sol(E)$ *in* F^4 *for*

$$E: \quad x - y + 2z + t = 0.$$

Exercise 0.6. *Find a basis for each of the subspaces* $\ker L_A$ *and* $\mathrm{im}\, L_A$ *of* F^4*, where* A *is the matrix*

$$\begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & -2 & 4 & 2 \\ 1 & 0 & 1 & 1 \\ 3 & 4 & -5 & -1 \end{bmatrix}.$$

Exercise 0.7. *We know that* $V^2 = V \times V$ *form a vector space. Define an* F*-vector space structure on* $U \times V$ *a vector space. Let's call it the* **direct sum** $U \oplus V$ *of* U, V*. If* $\dim U = k$ *and* $\dim V = n$*, what is* $\dim(U \oplus V)$*? Prove your assertion in 5 lines.*

Exercise 0.8. *(Revisit MMC) We specialize to the case* $V = F^2$*. Let* $(v_1, v_2) \in V^2$*, put* $A = [v_1, v_2] \in M_{2,2}$*, and write* $v_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$*,* $v_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$.
(a) (A numerical test for isomorphism) Show that v_1, v_2 *are 'parallel', i.e. one a scalar multiple of the other, iff they are dependent, iff*

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

iff L_A *is not an isomorphism, iff* (v_1, v_2) *is not a basis of* V.
(b) Now suppose L_A *is an isomorphism. Can you find the matrix* B *corresponding to (under MMC) to the inverse isomorphism* $L_A^{-1} : F^2 \to F^2$*?*

Exercise 0.9. *WRITE UP Let* F *be a field and* E *a subfield of* F.
(a) Show how you can make F *an* E*-space.*
(b) Assume that $F = E(\alpha)$ *where* $\alpha \in F$ *satisfies a polynomial equation*

$$p(\alpha) = 0$$

of degree $d \geq 0$*, and that* d *is the smallest such integer. Prove that* $\dim_E F = d$ *by giving a* E*-basis of* F.
(c) Let V *be an* F*-space. Show how you can make* V *an* E*-space.*
(d) Compute $\dim_E V$ *in terms of* $\dim_E F$ *and* $\dim_F V$ *which you can assume both finite.*