# LINEAR ALGEBRA HOMEWORK 4

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Assume $U, V, W$ are $F$-vector spaces. Put $\operatorname{End} V = \operatorname{Hom}(V, V)$.

**Exercise 0.1.** *Find the dimension of $M_{2,2}$ by giving a basis of this vector space. Generalize your result to $M_{k,l}$.*

**Exercise 0.2.** *Let $f, g : V \to V$ be two given maps such that $f \circ g = id_V$.*
*(a) Show that $g$ is injective and $f$ is surjective.*
*(b) Assume in addition that $\dim V < +\infty$ and $f$ is linear. Show that $f$ is injective, hence $g$ is surjective. (Hint: Use COD.)*
*(c) Conclude that $g$ is bijective, and that $f = g^{-1}$ and $g \circ f = id_V$.*
*(d) Let $A, B \in M_{n,n}$. Show that if $AB = I$, then $BA = I$.*
*(e) Second proof. Show that if $\ker(BA) = (0)$ then $\ker A = (0)$, hence $A$ is an isomorphism. Conclude that $B = A^{-1}$. (Hint: COD.)*

**Exercise 0.3.** WRITE UP *Prove that for $A \in M_{n,n}$, $\det A^t = \det A$. You will need the fact that $\operatorname{sgn} \sigma^{-1} = \operatorname{sgn} \sigma$ for any bijection of $\{1, 2, .., n\}$.*

**Exercise 0.4.** *Decide if $A = [e_3, e_1 + e_2, e_2] \in M_{3,3}$ is invertible. If so, compute $A^{-1}$. Here $e_i$ are the standard unit vectors if $F^3$.*

**Exercise 0.5.** *Let $U \subset V$ be a subspace and $x \in \operatorname{End} V$ such that $xU \subset U$. In 5 lines, prove that there is a canonical map*

$$\bar{x} : V/U \to V/U, \quad v + U \mapsto xv + U.$$

*That is check that this is well-defined. Show it satisfies the following: if $p(t) \in F[t]$, and $p(x) = 0$ in $\operatorname{End} V$ then $p(\bar{x}) = 0$ in $\operatorname{End} V/U$.*

**Exercise 0.6.** *By row reduction, compute*

$$\det[e_3 + e_2 + e_1, e_1 + e_2, e_2].$$

*Redo this it by using linearity of $\det$ in each column.*

**Exercise 0.7.** *Assume that $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}$ is invertible. Find a formula for $A^{-1}$. That is to say, find each entry of $A^{-1}$ in terms of the 4 entries $a_{ij}$ of $A$. Be sure to check that you do get $AA^{-1} = A^{-1}A = I$. From this, can you guess the answer for $3 \times 3$ matrices.*

**Exercise 0.8.** WRITE UP *Prove that the minimal polynomial of a matrix $A \in M_{n,n}$ is conjugation invariant, i.e. $\mu_{g^{-1}Ag}(x) = \mu_A(x)$ for all $g \in \operatorname{Aut}_n$. Conclude that the algebra $F[x]/\mu_A(x)F[x]$ does not change under conjugations of $A$.*

**Exercise 0.9.** *Compute the $\mu_A(x)$ for $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.*

**Exercise 0.10.** *WRITE UP For a given* $A \in M_{m\times n}$, *propose an algorithm to compute a basis of* $\ker A$ *and* $\operatorname{im} A$ *by row operations. In 10 lines prove that your algorithm is correct.*