2022 MATHCAMP RESEARCH PROJECTS: LINEAR ALGEBRA

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This is a preliminary version of the research projects, subject to change later.

0. Basic Assumptions and Notations

Unless stated otherwise, we shall make the following assumptions and use the following notations. F will denote a field of characteristic zero (i.e. F contains $\mathbb{Z}$ as a subset). For simplicity, you can think about the case $F = \mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers. A vector space means a finite dimensional F-vector space, usually denoted by $U, V, W, ...$ Likewise a linear map means an F-linear, and an F-matrix means a matrix with entries in F. Put

$$\begin{aligned}
\mathrm{Hom}(U,V) &:= \{\text{linear maps } U \to V\} \\
\mathrm{End}\, V &:= \mathrm{Hom}(V,V), \ \ \text{the algebra of linear maps } V \to V \\
\mathrm{Aut}\, V &:= \{f \in \mathrm{End}\, V | f \text{ is bijective}\} \\
\mathrm{Aut}\,_n F &:= \mathrm{Aut}\, F^n \\
(M_n, \times) \equiv M_n \equiv M_{n,n}(F) &:= \text{the associative algebra of } n \times n \ F\text{-matrices} \\
& \ \text{with the usual matrix product} \\
I &\equiv I_n := [e_1, .., e_n], \ \ \text{the identity matrix in } M_n
\end{aligned}$$

We usually denote composition of maps as $fg \equiv f \circ g$.

These objects will be quite thoroughly studied in class during the first two weeks.

1. Statements of Problems in Project 1

Problem 1.1.
(a) Describe all possible solutions to the matrix equation system

$$x_1^2 = x_1, \quad x_2^2 = x_2, \quad x_1 x_2 = x_2 x_1 = 0$$

in two variables in M_2, up to conjugation by Aut_2.
(b) Describe all those conjugation classes that satisfy the additional equation

$$x_1 + x_2 = I_2.$$

(c) Describe all possible two-sided ideals of M_2.

We saw in class that the algebra M_n itself is an M_n-space on which M_n acts by left multiplication. We also saw that an F-subspace $W \subset M_n$ is an M_n-subspace iff W is a left ideal of M_n.

Problem 1.2.
(a) Describe all possible left ideals I of M_2. Which ones of them are isomorphic to each other?
(b) Classify all minimal M_2-spaces V up to isomorphisms.
(c) Classify all M_2-spaces V up to isomorphisms.

Problem 1.3. *Generalize both Problems 1.1 and 1.2 to M_n-spaces for all n.*

2. Statements of Problems in Project 2

All graphs are assumed finite (i.e. have finite number of vertices), planar (i.e. you can draw on the plane) and oriented (i.e. every edge has a direction). All F vector spaces considered here are finite dimensional. A face of a graph L is a free region (i.e. no edges crosses it) bounded by edges of L. Let L be a graph, and $\mathcal{V}_L$, $\mathcal{E}_L$, $\mathcal{F}_L$ be the sets of vertices, edges, and faces of L respectively. We drop L if there is no confusion.

Every face $\phi \in \mathcal{F}$ is given the counter-clockwise orientation. Note that we can label ϕ by giving the list of edges in counterclockwise order. This induces an orientation on each edge e of ϕ which may or may not agree with the *given* orientation of e. We assign $+1$ to e and write $(\pm)_{\phi,e} = +1$ if the orientation induced by ϕ on e is the same as the given orientation of e. Otherwise we assign $(\pm)_{\phi,e} = -1$. We can label each $e \in \mathcal{E}$ by its vertices, say (a, b), where a is the initial and b is the end vertices of e. For convenience, we treat $(b, a) \equiv -(a, b)$ for each oriented edge $(a, b) \in \mathcal{E}$.

This project is about studying certain connections between graphs and F-vector spaces. Given such a graph L, we shall form a collection of vector spaces as follows:

- $C_0(L)$ is the F vector space given by taking the set of vertices of L as a basis of $C_0(L)$, i.e. $C_0(L) = F\mathcal{V}$.
- $C_1(L)$ is the F vector space given by taking the set of edges of L as a basis of $C_1(L)$, i.e. $C_1(L) = F\mathcal{E}$.
- $C_2(L)$ is the F vector space given by taking the set of faces of L as a basis of $C_2(L)$, i.e. $C_2(L) = F\mathcal{F}$.

This collection $C_\bullet(L)$ of vector spaces is called the **complex** of L.

We define a collection of linear maps, we call the boundary maps of L, between these vector spaces as follows. Write $C_i = C_i(L)$. Define

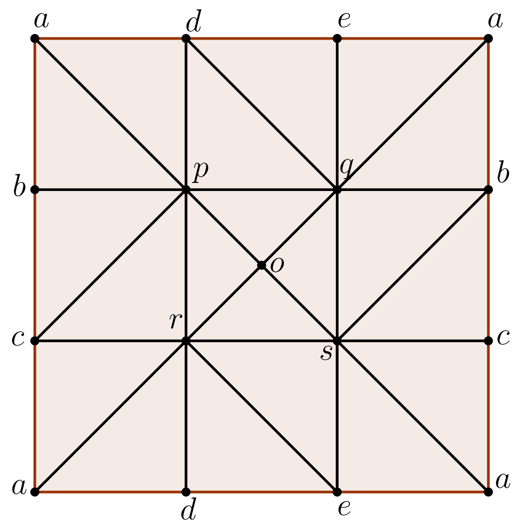
- $\partial_3 : (0) \to C_2$ is the zero map.
- $\partial_2 : C_2 \to C_1$, $\partial\phi = \sum(\pm)_{\phi,e}e$, where the sum is over all edges of ϕ.
- $\partial_1 : C_1 \to C_0$, $\partial(a, b) = a - b$.
- $\partial_0 : C_0 \to (0)$ is the zero map.

If there is no confusion, we shall drop all subscript from ∂.

Exercise 2.1. *Convince yourself that $\partial^2 = 0$, i.e. $\partial_i\partial_{i+1} = 0$ for all i. Therefore, we can define the* **homology** *spaces of L as*

$$H_i(L) := \ker\partial_i/\operatorname{im}\partial_{i+1}, \quad i = 0.1.2.$$

In this project, you are asked to determine the dimensions of the homology spaces of certain graphs. Consider the following graph, denoted by L.

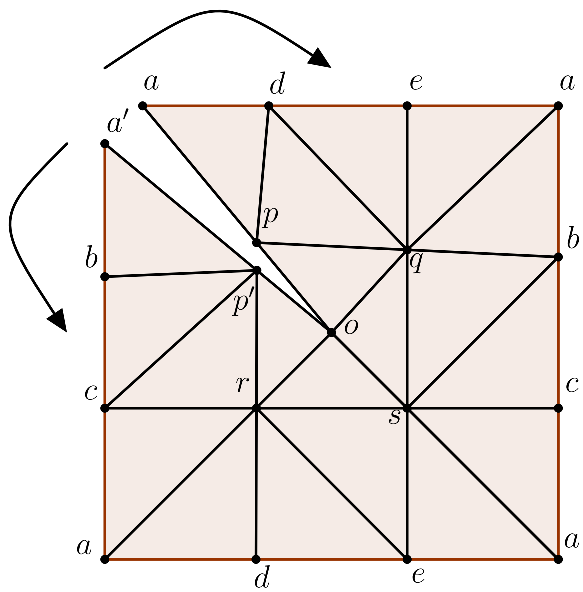


The boundaries of L (the brown edges) are identified via the labeling. We pick an orientation for every edge. For example, we can use the lexicographic ordering the vertex labels to give a edge an orientation: the edge with label (a, d) will have direction a to d, because a comes before d in the lexicographic ordering.

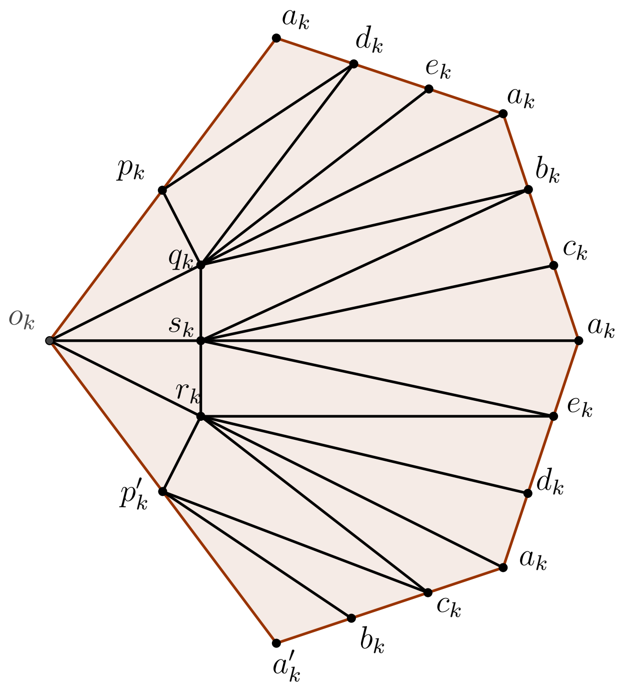
Problem 2.2. *Verify that*

- $\mathrm{H}_0(L) \simeq F$,
- $\mathrm{H}_1(L) \simeq F^2$,
- $\mathrm{H}_2(L) \simeq F$.

Next, we shall construct more complicated graphs by 'gluing' together copies of the graph L above. First, we 'cut open' L along the line segment $\overline{opa}$ in L, and then relabel the vertices as follows to get a new graph A_k.



Let A_k be the graph obtained by 'opening up' the shape along the arrows, and then assign the following new labels on the vertices.



For $n \in \mathbb{N}$, consider the set of graphs $\{A_1, \ldots, A_n\}$. Let S_n be the graph obtained by identifying $\overline{o_k p_k a_k}$ with $\overline{o_{k-1} p'_{k-1} a'_{k-1}}$ for $1 \leq k \leq n$. Here $\overline{o_0 p'_0 a'_0}$ is understood as $\overline{o_n p'_n a'_n}$. We orient each face in S_n counterclockwise and each edge in S_n, by lexicographic ordering as before. Again we get $C_\bullet(S_n)$ of the graph S_n. Note that $S_1 = L$.

Problem 2.3. *Find the homology spaces* $\mathrm{H}_q(S_n)$, *for all* n.

You should prove your answer.

3. Statements of Problems in Project 3

All vector spaces in this project are assumed finite dimensional.

A **complex** is a sequence of F-linear maps

$$\partial_{i+1} : C_{i+1} \to C_i, \quad i = 0, ..., d-1$$

such that the successive compositions are all zero: $\partial_i \partial_{i+1} = 0$. For convenience, we shall always assume that $C_i := (0)$, and that $\partial_{i+1} : C_{i+1} \to C_i$ are zero for all $i < 0$ or $i > d-1$. We denote the complex by $(C_\bullet, \partial)$ or simply $C_\bullet$, if ∂ is clear. The ith **homology** space of $C_\bullet$ is defined as the quotient spaces

$$H_i(C_\bullet) := \ker \partial_i / \operatorname{im} \partial_{i+1}, \quad i \in \mathbb{Z}.$$

A chain map between two complexes $(C_\bullet, \partial)$ and $(D_\bullet, \delta)$ is a collection of linear maps

$$f_i : C_i \to D_i$$

such that $\delta_i f_i = f_{i-1} \partial_i$ for all i. We shall denote the chain map by $f_\bullet : C_\bullet \to D_\bullet$.

We say that the two complexes are **equivalent** if there is a chain map $f_\bullet$ as above, such that each f_i is linear and bijective. We say that the two complexes are **quasi-equivalent** if there are two chain maps $f_\bullet : C_\bullet \to D_\bullet$, $g_\bullet : D_\bullet \to C_\bullet$, not necessarily bijective, such that their induced maps $\bar{f}_\bullet$, $\bar{g}_\bullet$ (to be defined in class) on homology are both linear and bijective, and are inverses of one another.

Problem 3.1. *Classify all equivalence classes of complexes $C_\bullet$ with at most two terms (i.e. all but two C_i are zero spaces). Do the same for 3-term complexes.*

Problem 3.2. *Classify all equivalence classes of complexes $C_\bullet$.*

Problem 3.3. *Classify all quasi-equivalence classes of complexes $C_\bullet$.*

4. Statements of Problems in Project 4

This is an open problem. You are encouraged to do some research online about the background of this problem. Be sure to give accurate references to whatever relevant information you have found.

Problem 4.1. *Given two commuting complex matrices* A, B *of size* $n \times n$*, when are they both polynomials of a third matrix* X*? That is to say, give a criterion such that there exists a pair of polynomials* $p(t), q(t) \in \mathbb{C}[t]$ *and a matrix* X *such that*

$$A = p(X), \quad B = q(X).$$

Problem 4.2. *Classify all pairs* (A, B)*, up to conjugations by* Aut_n*, of complex* $n \times n$ *commuting matrices:*

$$AB = BA.$$