

Homework 5/ Algebraic combinatorics

- (1) Let $G = S_3$ act on the 3-dimensional vector space $V = \langle e_1, e_2, e_3 \rangle$ by permuting the basis elements. Let $W = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1 + a_2 + a_3 = 0\}$ be a subrepresentation.
 - (a) If the ground field is \mathbb{C} , prove that W is irreducible.
 - (b) If the ground field is \mathbb{F}_3 , show that there is no subrepresentation W' such that $V = W \oplus W'$.
- (2) Let A_n be the alternating subgroup of S_n , consisting of all the even permutations.
 - (a) Prove that $|S_n|/|A_n| = 2$.
 - (b) Prove that A_n is a normal subgroup of S_n .
 - (c) Prove that the only 1-dimensional representations of S_n are the trivial representation, and the sign representation. (Hint: show that any group element of the form $aba^{-1}b^{-1}$ acts trivially on any 1-dimensional representation, and use this to show that A_n has to act trivially on any 1-dimensional representation of S_n .)