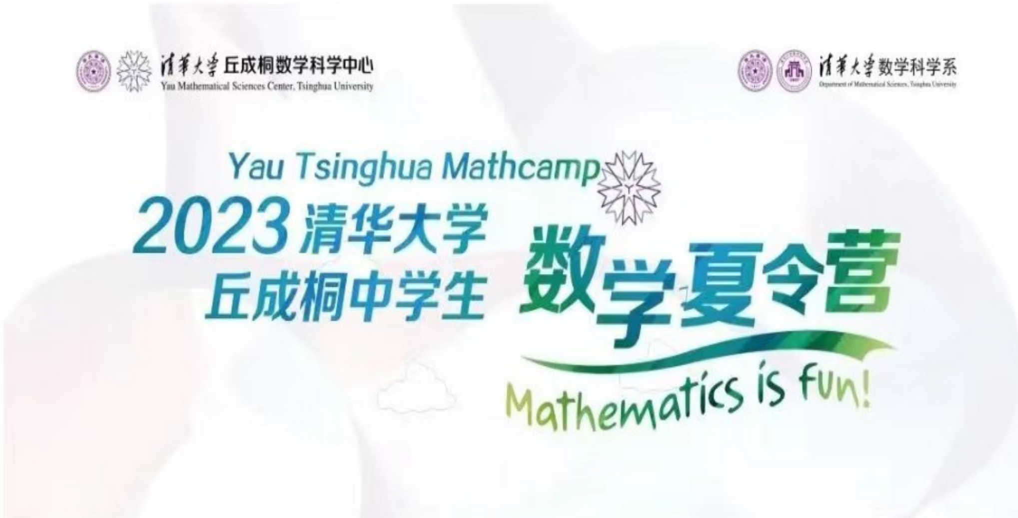

ANALYSIS AND NUMBER THEORY
SUMMER 2023
HOMEWORK NO. 1

1. Problems

Problem 1. Let $a_1, a_2, \ldots, a_n > 0$ and $b_1, b_2, \ldots, b_n > 0$. Show that

$$\sqrt[n]{\prod_{i=1}^{n} a_i} + \sqrt[n]{\prod_{i=1}^{n} b_i} \leq \sqrt[n]{\prod_{i=1}^{n} (a_i + b_i)}$$

Problem 2. Show that the sequence $(f_n)_{n\geq 1}$ defined by $f_n = \left(1 + \frac{1}{n}\right)^{n+1}$ is decreasing.

Problem 3. Show that if $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, then

$$n(n+1)^{1/n} < n + H_n, n > 1,$$

and

$$(n-1)n^{-1/(n-1)} < n - H_n, n > 2.$$

Problem 4. Let $e_n = \sum_{k=0}^{n} \frac{1}{k!}$, $n \geq 1$. Consider the sequence $E_n = e_n + \frac{1}{n!n}$. Show that $E_n > e$ and deduce that e is irrational.

Problem 5. Find the following limits:

(i) $\lim_{n\to\infty} n\left(\log 2 - \sum_{k=1}^{n} \frac{1}{n+k}\right)$

(ii) $\lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n}\sqrt{1+\frac{1}{n+k}}\right)^{n}$.

Problem 6. Let $(a_n)_{n\geq 1}$ be a sequence of positive reals such that $\sum_{n\geq 1} a_n^3$ converges. Show that the series $\sum_{n\geq 1} \frac{a_n}{n}$ converges also.

Problem 7. Let $0 < x_1 < 1$ and $x_{n+1} = x_n(1-x_n)$, for $n = 1, 2, 3 \ldots$. Show that $\lim_{n\to\infty} nx_n = 1$ and study the convergence of the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} x_n^2$.

Problem 8. Let $(a_n)_{n\geq 1}$ be a decreasing sequence of positive reals such that the series $\sum_{n=1}^{\infty} a_n$ converges. Show that $\lim_{n\to\infty} na_n = 0$.

Problem 9. Show that if the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ is convergent, where $p_1, p_2, \ldots, p_n$ are positive real numbers, then the series

$$\sum_{n=1}^{\infty} \frac{n^2}{(p_1 + p_2 + \ldots + p_n)^2} \cdot p_n$$

is also convergent.

Problem 10. Determine, with proof, whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.8+\sin n}}$$

converges or diverges.