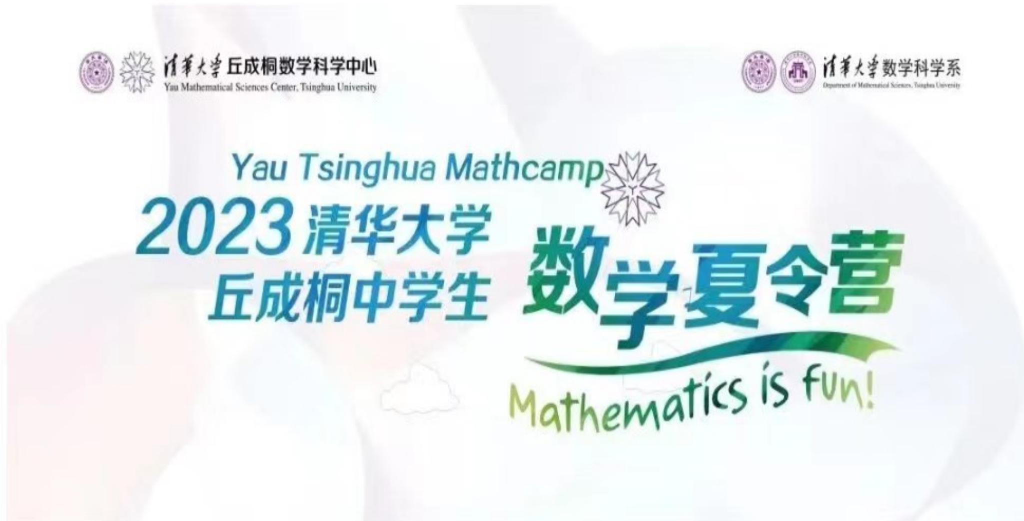

ANALYSIS AND NUMBER THEORY
SUMMER 2023
HOMEWORK NO. 2

1. Problems

Problem 1. Let $f(x) = \sum_{k=1}^{n} a_k \sin(kx)$, with $a_1, a_2, \ldots, a_n \in \mathbb{R}$, $n \geq 1$. Prove that if $f(x) \leq |\sin x|$, for all $x \in \mathbb{R}$, then

$$\left| \sum_{k=1}^{n} k a_k \right| \leq 1.$$

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements.

(a) If f is continuous and range(f)=$\mathbb{R}$ then f is monotonic.
(b) If f is monotonic and range(f)=$\mathbb{R}$ then f is continuous.
(c) If f is monotonic and f is continuous then range(f)=$\mathbb{R}$

Problem 3. (a) If f is a C^2 function on an open interval, prove that

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

(b) Suppose that $f(x)$ is defined on an open interval containing x, and $f(x)$ is three times differentiable on this interval. Show that

$$\lim_{h \to 0} \frac{f(x+h) - 3f(x) + 3f(x-h) - f(x-2h)}{h^3} = f'''(x).$$

Problem 4. (i) Show that $e^x \geq x+1$ for each $x \in \mathbb{R}$, with equality if and only if $x = 0$.

(ii) Let $a_k > 0$ for $k = 1, 2, \ldots, n$, and let $A = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$ be the arithmetic mean of these numbers. For each k, put $x_k = \frac{a_k}{A} - 1$ in the inequality from (i), and deduce the arithmetic-geometric mean inequality:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

Problem 5. (i) Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist and they are both finite. Show that $\lim_{x\to\infty} f'(x) = 0$.

(ii) Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $\lim_{x\to\infty} (f(x) + f'(x)) = 0$. Show that $\lim_{x\to\infty} f(x) = 0$.

Problem 6. Let f be an infinitely differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that, for some positive integer n,

$$f(1) = f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = 0.$$

Prove that $f^{(n+1)}(x) = 0$ for some $x \in (0, 1)$.

Problem 7. Let α be a real number. Show that there does not exist a continuous real-valued function $f : [0, 1] \to (0, \infty)$ such that

$$\int_0^1 f(x)dx = 1, \int_0^1 xf(x)dx = \alpha, \int_0^1 x^2 f(x)dx = \alpha^2.$$

Problem 8. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that $f(0) = f(1) = 0$. Show that

$$\int_0^1 (f'(x))^2 dx \geq 12 \left(\int_0^1 f(x)dx \right)^2.$$

Problem 9. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x+1) = f(x)$ and $g(x+1) = g(x)$ for all real numbers x. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

Problem 10. (a) Consider the sequence $(x_n)_{n\geq 1}$ defined by

$$x_n = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \ldots + (-1)^n \frac{1}{n^2}.$$

Show that x_n converges and assuming The Basel problem,

$$\lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)=\frac{\pi^2}{6},$$

show that $\lim_{n\to\infty} x_n = \dfrac{\pi^2}{12}$.

(b) Show that the following version of Taylor's formula with integral remainder holds true:

$$\log(1+x)=x-\frac{x^2}{2}+\ldots+(-1)^{n-1}\frac{x^n}{n}+(-1)^n\int_0^x\frac{t^n}{1+t}dt,$$

for all $n\geq 1$ and $x>-1$.

(c) Show that the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}\dfrac{\log(1+x)}{x}, & x\in(0,1]\\ 1, & x=0\end{cases}$$

is Riemann integrable.

(d) Show that

$$\int_0^1\frac{\log(1+x)}{x}dx=\frac{\pi^2}{12}.$$