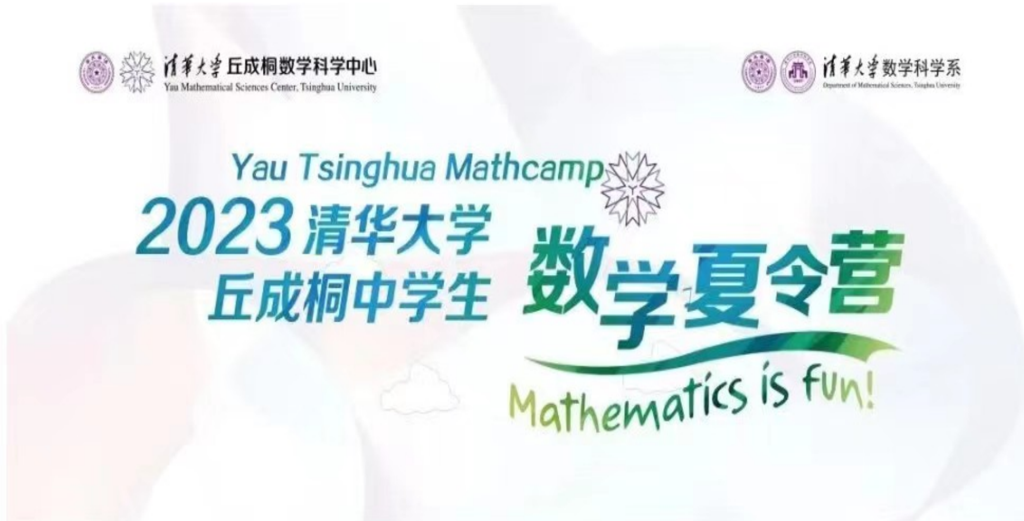

ANALYSIS AND NUMBER THEORY
SUMMER 2023
HOMEWORK NO. 3

1. Problems

Problem 1. Show that:

(i) $\frac{6}{\pi^2} < \frac{\sigma(n)\varphi(n)}{n^2} < 1, n \geq 1.$

(ii) $\sigma(n)\varphi(n) \geq nd(n), n \geq 3.$

(iii) $\sigma^2(n) \geq nd^2(n), n \geq 1.$

Problem 2. Show that for $x \geq 2$,

$$\sum_{n \leq x} \frac{1}{n \log n} = \log\log x + B + O\left(\frac{1}{x \log x}\right),$$

where B is a constant.

Problem 3. Show that for $x \geq 3$ we have

$$\sum_{n \leq x} (\log n)^2 = x \log^2 x - 2x \log x + 2x + O(\log^2 x).$$

Problem 4. Show that

$$\sum_{i,j=1}^{n} \gcd(i,j) = \frac{n^2 \log n}{\zeta(2)} + O(n^2).$$

Problem 5. Find all positive integers $n \geq 1$ such that $d(n) = \sqrt[3]{4n}$.

Problem 6. Find $x \in (0,1)$ such that

$$\sum_{n=1}^{\infty} \frac{x^n \varphi(n)}{1 - x^n} = 1.$$

Problem 7. Let $(a_n)_{n\geq 1}$ be an increasing sequence such that $a_{n+1} - a_n < 2023$ for all $n \geq 1$. Show that the set of prime factors of the sequence a_n is infinite.

Problem 8. Given a positive integer a, prove that $\sigma(am) < \sigma(am+1)$ for infinitely many positive integers m.

Problem 9. Let k be a fixed integer greater than 1. Prove that there exists an integer n greater than 1, and distinct integers $a_1, a_2, \ldots, a_n$ all greater than 1, such that both $\sum_{j=1}^{n} a_j$ and $\sum_{j=1}^{n} \phi(a_j)$ are kth powers of a positive integer. Here ϕ denotes the Euler's totient function.

Problem 10. For a positive integer n, let $d(n)$ be the number of positive divisors of n, and let $\varphi(n)$ be the number of positive integers not exceeding n which are coprime to n. Does there exist a constant C such that

$$\frac{\varphi(d(n))}{d(\varphi(n))} \leq C$$

for all $n \geq 1$?