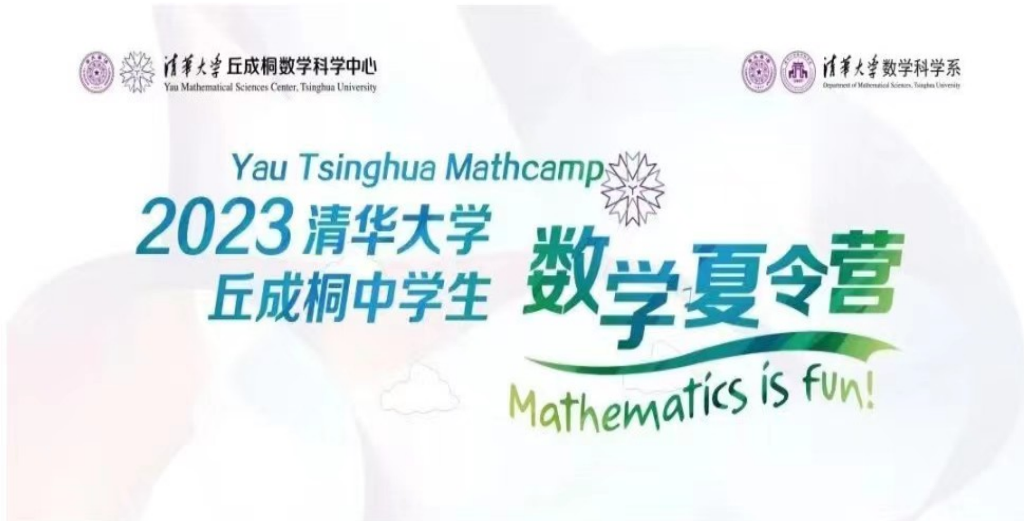

ANALYSIS AND NUMBER THEORY
SUMMER 2023
HOMEWORK NO. 4

1. Problems

Problem 1. Show that the harmonic sequence $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is not an integer for all $n \geq 1$.

Problem 2. Show that

$$\pi(n! + 2n) + \pi(n) \leq \pi(n! + n) + \pi(2n),$$

for all $n \geq 1$.

Problem 3. Show that

$$\sum_{k=1}^{n} \frac{p_{k+1} - p_k}{\log p_k} \sim n, n \to \infty.$$

Problem 4. Let p_n be the nth prime number. Show that the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{p_{n+1}} - \sqrt{p_n}}{n}$$

converges.

Problem 5. Show that $n^{\pi(2n)-\pi(n)} < 4^n$ for all $n > 1$. Deduce that there is a constant $c > 0$ such that

$$\pi(n) \leq c \cdot \frac{n}{\log n}.$$

Problem 6. Show that $\pi(n)$ divides n for infinitely many n.

Problem 7. Define

$$f(m,n) = \frac{1}{mn^3} + \frac{1}{2m^2n^2} + \frac{1}{m^3n}.$$

(i) Show that

$$f(m,n) - f(m+n,n) - f(m,m+n) = \frac{1}{m^2n^2}.$$

(ii) Assuming that $\zeta(2) = \frac{\pi^2}{6}$ show that $\zeta(4) = \frac{\pi^4}{90}$.

Problem 8. For positive integers $k_1, k_2, \ldots, k_r$ with $k_r > 1$, define the nested series called *multiple zeta values*,

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{1 \leq n_1 < n_2 < \ldots < n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}.$$

Show that the above series defines a real number.

Problem 9. Show that $\zeta(2,2) = \frac{\pi^4}{120}$.

Problem 10. Show that $\zeta(1,2) = \zeta(3)$.