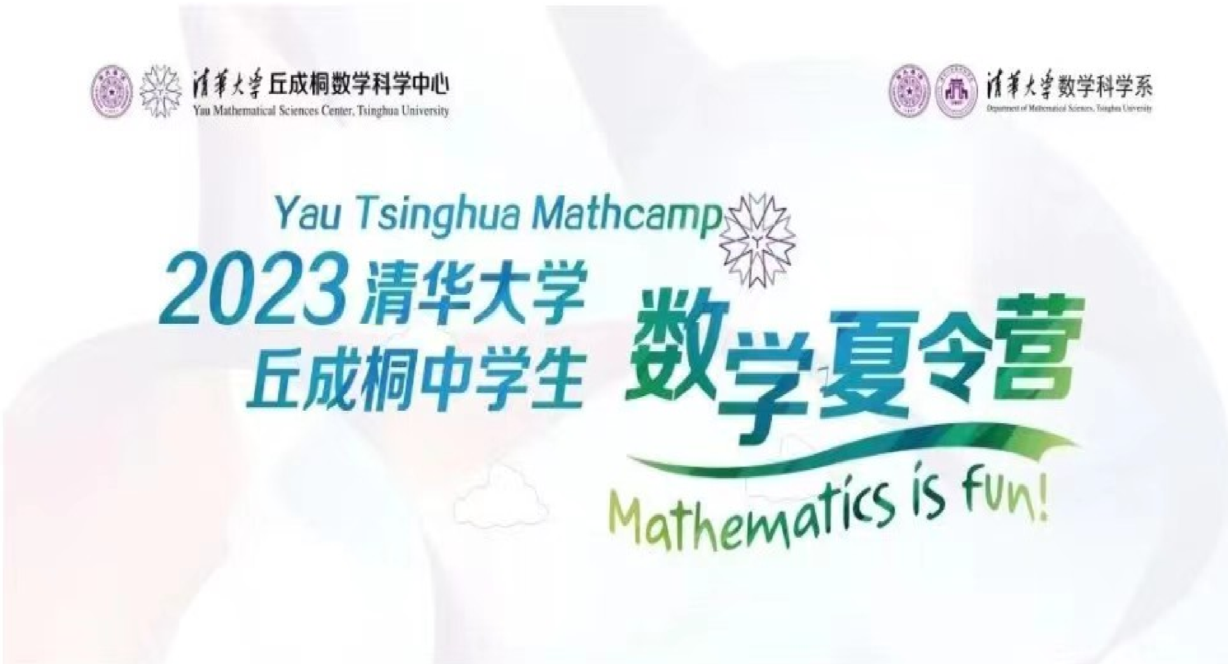

ANALYSIS AND NUMBER THEORY SUMMER 2023 LIST OF PROJECTS

CEZAR LUPU

1. An algebraic equation and Euler's Basel problem

Project. (a) Show that the following algebraic equation of degree n,

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \binom{2n+1}{5}x^{n-2} - \ldots = 0$$

has solutions $x_k = \cot^2 \dfrac{k\pi}{2n+1}$, for $k = 1, 2, \ldots, n$.

(b) Show that

$$\frac{1}{\sin x} > \frac{1}{x} > \cot x, x \in \left(0, \frac{\pi}{2}\right).$$

(c) Prove the inequality

$$\frac{\pi^2}{6}\left(1 - \frac{1}{2n+1}\right)\left(1 - \frac{2}{2n+1}\right) < 1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} < \frac{\pi^2}{6}\left(1 - \frac{1}{2n+1}\right)\left(1 + \frac{1}{2n+1}\right).$$

(d) Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. Wallis' integral formula and Euler's Basel problem

Project. (a) Write the binomial series expansion of $(1-x^2)^{-1/2}$ near $x=0$.
(b) Derive the Taylor series expansion of $\arcsin x$ near $x=0$,

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1\cdot 3 \ldots \cdot (2n-1)}{2\cdot 4 \ldots \cdot (2n)} \cdot \frac{x^{2n+1}}{2n+1}.$$

(c) Explain why the series above converges uniformly on the interval $[-1,1]$.
(d) Derive the following equality

$$\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{1}{2n+1} \cdot \frac{1\cdot 3 \cdot \ldots \cdot (2n-1)}{2\cdot 4 \cdot \ldots \cdot (2n)} \int_0^{\frac{\pi}{2}} \sin^{2n+1} t dt.$$

(e) Prove the following Wallis integral formula:

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} t dt = \frac{2\cdot 4 \cdot \ldots \cdot (2n)}{1\cdot 3 \cdot \ldots \cdot (2n-1)} \cdot \frac{1}{2n+1}, n \geq 1.$$

(f) Derive Euler's famous formula,

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. Power series and Euler's Basel problem

Project. The Basel problem is one of the most famous problems in analysis and number theory and it concerns the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

The result was proved by Euler in 1734 and is given by

Theorem 3.1 (Euler). *We have*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this project you will be guided to obtain a proof of this result via generating functions.

First, **prove** that the function $y(x) = \arcsin^2 x$ verifies the second order initial value problem,

$$(1-x^2)y'' - xy' - 2 = 0, y(0) = y'(0) = 0.$$

Second, when we look for the power series solution of the above initial value problem, $y(x) = \sum_{n\geq 0} a_n x^n$, **prove** that

$$\arcsin^2 x = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2\binom{2n}{n}}, |x| \leq 1.$$

Third, with a little bit of extra care, **prove** the convergence of the series above.
Fourth, **prove** the following Wallis' formula,

$$\int_0^{\frac{\pi}{2}} \sin^{2n} t dt = \frac{\pi}{2^{2n+1}}\binom{2n}{n}.$$

Last but not least, combine all these previous steps, and **derive** the proof of the Basel problem.

4. Convex functions, Hermite-Hadamard inequality, and Stirling's approximation formula

Project. Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. First, derive the *Hermite-Hadamard* integral inequality,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

We can proceed in the following way:
Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be a convex function.

(i) Show that for all $a \leq x \leq y \leq z \leq t$ with $x + t = y + z$ we have the inequality

$$f(x) + f(t) \geq f(y) + f(z).$$

(ii) Show that the function $g : [a,b] \to \mathbb{R}$, $g(x) = f(x) + f(a+b-x)$, $x \in [a,b]$ is nonincreasing on $\left[a, \frac{a+b}{2}\right]$, and nondecreasing on $\left[\frac{a+b}{2}, b\right]$.
(iii) Show that f is Riemann integrable.
(iv) Show that the function $h : [a,b] \to \mathbb{R}$ defined by $h(x) = f(a+b-x)$, $x \in [a,b]$ is convex and

$$\int_a^b f(x)dx = \int_a^b h(x)dx.$$

(v) Prove that the *Hermite-Hadamard* inequality holds true.
Furthermore, prove that:

(a) Show that

$$\int_k^{k+1} \log x \geq \frac{\log k + \log(k+1)}{2}, k \geq 1.$$

(b) Show that

$$\int_{k-1/2}^{k+1/2} \log x \leq \log k, k \geq 1.$$

(c) Consider the sequence

$$a_n = \int_1^n \log x - \log 2 - \ldots - \log(n-1) - \frac{1}{2}\log n, n \geq 1.$$

Show that a_n is increasing and $0 \leq a_n \leq \frac{1}{2}\log\frac{5}{4}$.

(d) Prove the following inequality:

$$e\sqrt{n}\left(\frac{n}{e}\right)^n \geq n! \geq \sqrt{\frac{4}{5}}e\sqrt{n}\left(\frac{n}{e}\right)^n, n \geq 1.$$

(e) Prove the following formula due to *Stirling*:

$$\lim_{n\to\infty} \frac{n!}{n^n \cdot e^{-n}\sqrt{2\pi n}} = 1.$$

5. Limits of sequences of functions and the irrationality of π

Project. Let $a, b, n \geq 1$ be integers, and consider the sequence of functions $f_n \to \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n!}x^n(bx-a)^n$, and the sequence of integrals $(I_n)_{n\geq 1}$ defined by

$$I_n = \int_0^\pi f_n(x)\sin x dx, n \geq 1.$$

(a) Show that if $u, v : I \to \mathbb{R}$ are functions n-times differentiable on I, then

$$(uv)^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(n-k)}v^{(k)}.$$

(b) Show that $f_n^{(k)}(0) \in \mathbb{Z}$ for all $k \geq 1$ and $n \geq 1$.
(c) Show that $f_n^{(k)}(\frac{a}{b}) \in \mathbb{Z}$ for all $k \geq 1$ and $n \geq 1$.
(d) Show that $\lim\limits_{n\to\infty} I_n = 0$.
(e) Show that if $\pi = \frac{a}{b}$, with $a, b \geq 1$ integers, then $I_n \in \mathbb{Z} - \{0\}$.
(f) Show that π is an irrational number.

6. An example of a function which is not differentiable anywhere: Weierstrass function

Project. Let us consider the function $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = \dfrac{1 - |2x - 2[x] - 1|}{2}$ and the functions $f_n(x) = 4^{-n+1} \cdot f_1(4^{n-1}x)$, $x \in \mathbb{R}$, $n \geq 1$.

(a) Show that $0 \leq f_1(x) \leq \frac{1}{2}$, for all $x \in \mathbb{Z}$ and $f_1(x+1) = f_1(x)$, for all $x \in \mathbb{R}$.
(b) Show that the function $f(x) = \sum_{n=0}^{\infty} f_n(x)$, $x \in \mathbb{R}$ is well defined and continuous.
(c) Show that the function f is not monotonic on any interval.
(d) Show that f is not differentiable at any point.

7. Evaluation of $\zeta(2)$ and the representation of a number as sum of squares

Project. Let $r(n)$ be the number of quadruples (x, y, z, t) of integers such that

$$n = x^2 + y^2 + z^2 + t^2.$$

(a) Show that $r(0) = 1$ and $r(n) = 8 \displaystyle\sum_{m|n, 4\nmid m} m, m > 0$.

(b) Let $R(N) = \sum_{n=0}^{N} r(n)$. Show that $R(N)$ is asymptotic to the volume of the 4-dimensional ball, i.e.

$$R(N) \sim \frac{\pi^2}{2} N^2.$$

(c) Evaluate $R(N)$ in terms of the function $\theta(x) = \displaystyle\sum_{m \leq x} m \left[\frac{x}{m}\right]$.

(d) Show that $\theta(x) = \dfrac{\zeta(2)}{2} x^2 + O(x \log x)$ and deduce that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

8. Erdös' proof of Bertrand's Postulate

Project.

(a) Let $n > 0$ and $r(p)$ be the non-negative integer such that

$$p^{r(p)} \leq 2n < p^{r(p)+1}.$$

Show that

$$\prod_{n<p\leq 2n} p \;\Big|\; \binom{2n}{n} \;\Big|\; \prod_{p \leq 2n} p^{r(p)}.$$

(b) Show that if $p > 2$ and $\frac{2n}{3} < p \leq n$, then

$$p \nmid \binom{2n}{n}.$$

(c) Show that

$$\prod_{p \leq n} p < 2^{2n}.$$

(d) Assume there is no prime p in between n and $2n$ ($n < p \leq 2n$). Prove that

$$2^{2n} < (2n)^{\sqrt{2n}+1} 2^{\frac{4}{3}n},$$

which is impossible for sufficiently large n. Hence there exists at least one prime in between n and $2n$ for sufficiently large n.

Applications.

(a) Find an upper bound for all the n's satisfying the inequality in Problem 8(d). Deduce that there is at least one prime in between n and $2n$ for any $n \geq 1$.

(b) Let p_n denote the n-th prime. Show that for $n > 3$,

$$p_n < p_1 + p_2 + \ldots + p_{n-1}.$$

9. Bernoulli polynomials and generalized Euler-Maclaurin summation formula

Project. We define the sequence of *Bernoulli polynomials* $B_n(x)$ and the *Bernoulli numbers* B_n as follows: we let $B_0(x) = B_0 = 1$, $B_1 = -1/2$ and $B_1(x) = x + B_1$. We then let $B_2(x) = B_2 + 2\int_0^x B_1(x)dx$, where B_2 is such that $\int_0^1 B_2(x)dx = 0$, that is to say, $B_2 = \frac{1}{6}$ and $B_2(x) = x^2 - x + 1/6$. In general, assuming we have defined $B_n(x)$, we let $B_{n+1}(x) = B_{n+1} + (n+1)\int_0^x B_n(t)dt$, where B_{n+1} is such that $\int_0^1 B_{n+1}(x)dx = 0$.

(a) For $n \neq 1$, show that $B_n(1) = B_n(0) = B_n$. Conclude that the function $x \mapsto B_n(\{x\})$ is 1-periodic and continuous. In addition, show that $\int_0^x B_n(\{t\})dt = (B_{n+1}(\{x\}) - B_{n+1})/(n+1)$ for all $n \geq 1$ and $x \in \mathbb{R}$.

(b) Given integers $a < b$ and $k \geq 1$, and a smooth function f, prove that

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + \sum_{l=1}^{k} \frac{(-1)^l B_l}{l!}(f^{(l-1)}(b) - f^{(l-1)}(a)) + (-1)^{k+1}\int_a^b \frac{B_k(\{x\})f^{(k)}(x)}{k!}dx.$$

(c) Let $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$. Show that

$$B_k(\{x\}) = -\frac{k!}{(2\pi i)^k}\sum_{m\neq 0}\frac{e^{2\pi i m x}}{m^k}, k \geq 2.$$

(d) For $k \geq 1$, show that $B_{2k+1} = 0$ and

$$B_{2k} = \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}}\sum_{m\geq 1}\frac{1}{m^{2k}} = \frac{(-1)^{k-1}(2k)!\zeta(2k)}{2^{2k-1}\pi^{2k}},$$

where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \mathrm{Re}(s) > 1$ is the Riemann zeta function.
(e) Show that for $n \geq 1$, we have

$$\sum_{n\leq N}\frac{1}{n} = \log N + \gamma + \frac{1}{2N} - \frac{1}{12N^2} + O(1/N^4).$$

10. An elementary problem equivalent to the Riemann hypothesis

Project. The Riemann hypothesis is one of the most important problems in mathematics concerning the non-trivial zeros of the zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \mathrm{Re}\, s > 1.$$

It states that all non-trivial zeros satisfy $\mathrm{Re}\, s = \frac{1}{2}$. In this aspect, there are many equivalent statements. One of them is given by the following:

$$(L) : \sigma(n) = \sum_{d|n} d \leq H_n + \exp(H_n)\log(H_n),$$

for all $n \geq 1$. Here H_n stands for the n-th harmonic number. The statement (L) is a modification of an earlier result of Robin which states that the Riemann hypothesis is equivalent with

$$\sigma(n) < e^{\gamma} n \log\log n,$$

for all $n \geq 5041$. Moreover, Robin was able to prove unconditionally that

$$\sigma(n) < e^{\gamma} n \log\log n + 0.6482\frac{n}{\log\log n}, n \geq 3.$$

Thus, we assume the following two results of Robin,

Theorem 10.1. *If the Riemann hypothesis is true, then for each $n \geq 5041$ we have*

$$\sigma(n) \leq e^{\gamma} n \log\log n,$$

where gamma is the Euler-Mascheroni constant.

and

Theorem 10.2. *If the Riemann hypothesis is false, then there exists constants* $0 < \beta < \frac{1}{2}$ *and* $C > 0$ *such that*

$$\sigma(n) \geq e^{\gamma} n \log\log n + \frac{Cn \log\log n}{(\log n)^{\beta}}$$

holds for infinitely many n.

Now, in order to prove that our inequality (L) is equivalent to the Riemann hypothesis, **you are asked to prove** the following:

Lemma 10.3. *For* $n \geq 3$, *we have*

$$\exp(H_n)\log(H_n) \geq e^{\gamma} n \log\log n.$$

Lemma 10.4. *For* $n \geq 20$, *we have*

$$H_n + \exp(H_n)\log(H_n) \leq e^{\gamma} n \log\log n + \frac{7n}{\log n}.$$

Finally, **deduce** that the inequality (L) is equivalent to the Riemann hypothesis.

11. Dirichlet's hyperbola method and applications

Project.

(a) Prove the *Dirichlet hyperbola formula.* Let f and g be two arithmetic functions and let $F(x) = \sum_{n \leq x} f(n)$ and $G(x) = \sum_{n \leq x} g(n)$. Show that for any $1 \leq y \leq x$,

$$\sum_{n \leq x} \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{n \leq y} f(n) G\left(\frac{x}{n}\right) + \sum_{m \leq x/y} g(m) F\left(\frac{x}{m}\right) - F(y) G\left(\frac{x}{y}\right).$$

(b) Prove that for $x \geq 1$,

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

(c)

$$\sum_{n \leq x} \left\{\frac{x}{n}\right\} = (1 - \gamma)x + O(\sqrt{x}),$$

where γ is Euler's constant.

(d) Show that

$$\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x).$$