2023 TSINGHUA MATHCAMP RESEARCH PROJECT
ANALYSIS AND TOPOLOGY 1
PROJECT A: PROJECTIVE PLANE AND ITS CORRESPONDING ALGEBRA STRUCTURE

You are encouraged to read through and have an understanding of these materials, while only focus on the problems that grab your interest.

1. Basic Notations

Definition 1.1. A *generalised plane* is a triple of sets $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ such that $\mathscr{I} \subseteq \mathscr{P} \times \mathscr{L}$. We call $\mathscr{P}$ the set of points, $\mathscr{L}$ the set of lines and $\mathscr{I}$ the set of incidence relations. If $(p, L) \in \mathscr{I}$, we say that L passes through p and denote it by $p \in L$.

Example. The triple $(\mathscr{P}_{\mathbb{R}^2}, \mathscr{L}_{\mathbb{R}^2}, \mathscr{I}_{\mathbb{R}^2})$ defined by

$$\mathscr{P}_{\mathbb{R}^2} := \{(x, y) \mid x, y \in \mathbb{R}\},$$
$$\mathscr{L}_{\mathbb{R}^2} := \{(a, b, c) \mid a, b, c \in \mathbb{R}, a^2 + b^2 \neq 0\},$$
$$\mathscr{I}_{\mathbb{R}^2} := \{(x, y) \times (a, b, c) \subseteq \mathscr{P}_{\mathbb{R}} \times \mathscr{L}_{\mathbb{R}} \mid ax + by + c = 0\}.$$

is a generalised plane (which contains more "lines" than the usual plane $\mathbb{E}^2$).

Example. Let $\mathscr{P}$ be the set of lines in $\mathbb{E}^3$ and $\mathscr{L}$ the set of planes in $\mathbb{E}^3$. We define $\mathscr{I}$ as

$$\mathscr{I} := \{(L, \alpha) \in \mathscr{P} \times \mathscr{L} \mid L \subset \alpha\}.$$

Then $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ is a generalised plane.

We want to study projective version of generalized planes:

Definition 1.2. A *projective plane* is a generalised plane $\pi = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ satisfy that

Pr1. Any two points lie on exactly one line;
Pr2. Any two lines pass through exactly one point;
Pr3. There exist 4 points so that no line passes through more than two of them.

Definition 1.3. Two projective planes $\pi = (\mathscr{P}, \mathscr{L}, \mathscr{I})$ and $\pi' = (\mathscr{P}', \mathscr{L}', \mathscr{I}')$ are isomorphic if there exist bijective maps $f : \mathscr{P} \to \mathscr{P}'$ and $F : \mathscr{L} \to \mathscr{L}'$ such that $p \in L$ if and only if $f(p) \in F(L)$. We denote it by $\pi \cong \pi'$.

Definition 1.4. A *division ring* (or a *skew field* in some of the books) is a set S equipped with two binary operations $+ : S \times S \to S$ and $\cdot : S \times S \to S$ satisfying:

- $+$ and $\cdot$ are associative, that is to say, $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

- There exist $o, i \in S$ satisfying $a + o = o + a = a$ and $a \cdot i = i \cdot a = a$ for all $a \in S$.
- For all $a \in S$, there exist $b \in S$ such that $a + b = b + a = o$. For all $a \in S \backslash o$, there exists $c \in S$ such that $a \cdot c = c \cdot a = i$.
- $+$ is commutative, that is to say, for all $a, b \in S$, $a + b = b + a$.
- $\cdot$ is distributive with respect to $+$, that is to say, for all $a, b, c \in S$, $(a + b) \cdot c = a \cdot c + b \cdot c$, and $a \cdot (b + c) = a \cdot b + a \cdot c$.

We often use letter D to denote a division ring.

Example. A field is always a division ring.

Example. Let $\mathcal{H} := \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}$, with $+$ and $\cdot$ defined as follow:

- $+$ is defined coordinatewise, that is to say, $(a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k) = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$
- $\cdot$ is defined according to the following table:

Right / Left	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

That is to say, $(a_1 + b_1 i + c_1 j + d_1 k) \cdot (a_2 + b_2 i + c_2 j + d_2 k) = (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2)i + (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2)j + (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)k$.

$\mathcal{H}$ is called *the ring of quaternions*. Checking the fact that it is a division ring is left to you as an exercise.

Definition 1.5. Two division rings $(D, +, \cdot)$ and $(D', \oplus, \odot)$ are isomorphic if there exists a bijective map $f : D \to D'$ such that f holds the operations, that is to say, $f(a + b) = f(a) \oplus f(b)$ and $f(a \cdot b) = f(a) \odot f(b)$. We denote it by $D \cong D'$.

2. Problem List

Problem 1. Let D be a division ring, and let $\mathscr{P}_D := D^3 - \{(0, 0, 0)\} / \sim_{\mathscr{P}}$ be the quotient space, where the equivalence relation is given by $(x, y, z) \sim_{\mathscr{P}} (x\lambda, y\lambda, z\lambda)$, $\forall \lambda \in D \neq 0$. (Keep in mind that the order of multiplication matters, and λ is on the **right**.)

We denote by $[x, y, z]$ for the class of $(x\lambda, y\lambda, z\lambda)$ where $\lambda \neq 0$. Clearly we have $[x, y, z] = [x\lambda, y\lambda, z\lambda]$.

Let $\mathscr{L}_D := D^3 - \{(0, 0, 0)\} / \sim_{\mathscr{L}}$,where the equivalence relation is given by $(a, b, c) \sim_{\mathscr{L}} (\mu a, \mu b, \mu c)$, $\forall \mu \in D \neq 0$. (Again, keep in mind that the order of multiplication matters. This time, μ is on the **left**.)

We denote by $\langle a, b, c \rangle$ for the class of $(\mu a, \mu b, \mu c)$ where $\mu \neq 0$. Clearly we have $\langle a, b, c \rangle = \langle \mu a, \mu b, \mu c \rangle$.

Let $\mathscr{I}_D := \{[x,y,z] \times \langle a,b,c\rangle \in \mathscr{P}_D \times \mathscr{L}_D \mid ax+by+cz=0\}$.

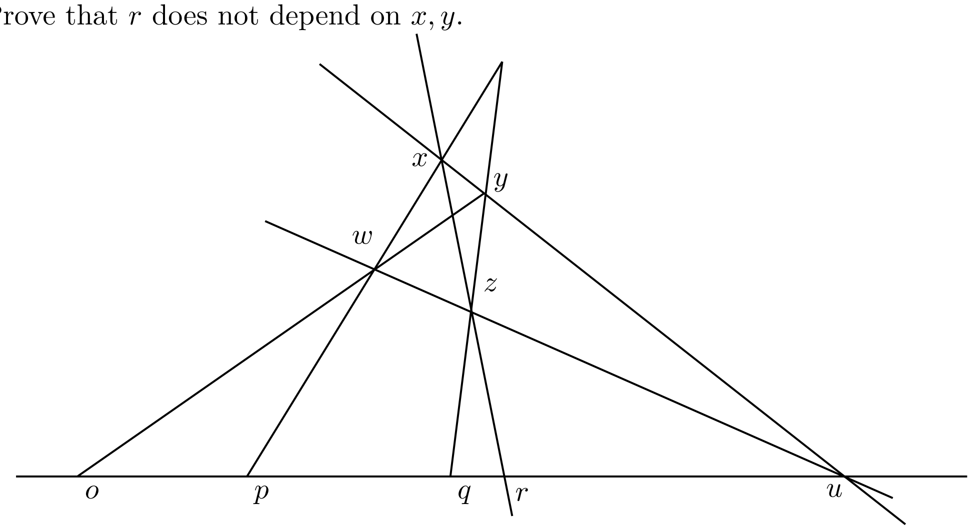
(1) Show that $\mathscr{I}_D$ is well defined. That is to say, $[x,y,z] \times \langle a,b,c\rangle \in \mathcal{I}_D$ holds or not does not depend on which element we choose from the equivalence class $[x,y,z]$ and $\langle a,b,c\rangle$.
(2) Draw a picture for $\pi(\mathbb{F}_2)$ and $\pi(\mathbb{F}_3)$, where for p prime, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{\overline{0}, \cdots, \overline{p-1}\}$,+ and $\cdot$ are defined modulo p.
(3) Show that $\pi(D)$ is a projective plane.
(4) Show that Desargues theorem holds on $\pi(D)$, and Pappus theorem holds on $\pi(D)$ if and only if D is a field.
(5) Construct a counterexample of Pappus theorem on $\pi(\mathcal{H})$.

Problem 2. Let π be a projective plane on which Desargues theorem holds.

Take two arbitary lines L and M in $\mathscr{L}$. Denote $u = L \cap M$ and let o, i be two arbitary distinct points on L different from u.

(1) For points p, q on L distinct from o and u (p and q need not to be distinct from each other), we construct another point r on L according to the following procedure:
- Step 1: Choose arbitary distinct points x, y on M distinct from u.
- Step 2: Connect px, qy and oy.
- Step 3: Denote $px \cap oy$ by w. Connect uw.
- Step 4: Denote $uw \cap qy$ by z. Connect xz.
- Step 5: Denote $xz \cap L$ by r.

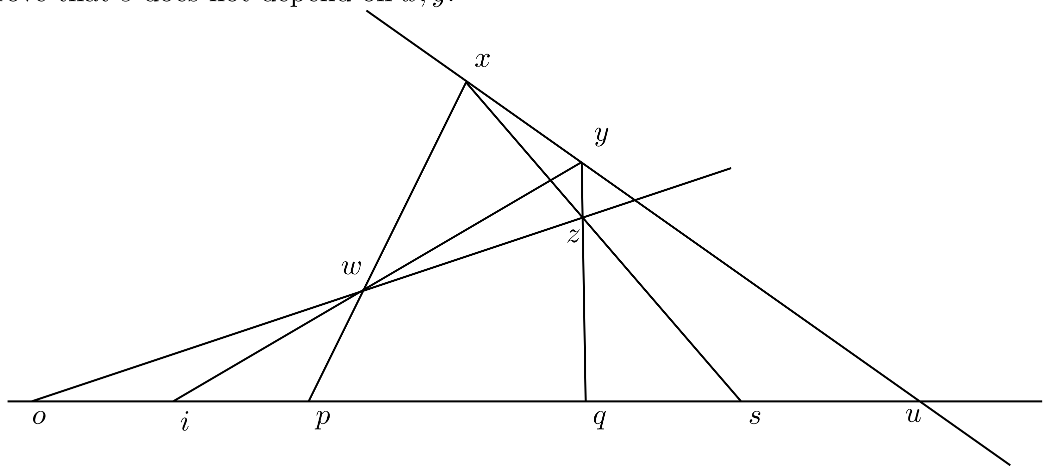
Prove that r does not depend on x, y.



(2) For points p, q on L distinct from o, e and u (p and q need not to be distinct from each other), we construct another point s on L according to the following procedure:
- Step 1: Choose arbitary distinct points x, y on M distinct from u.

- Step 2: Connect px, qy and iy.
- Step 3: Denote $px \cap iy$ by w. Connect ow.
- Step 4: Denote $ow \cap qy$ by z. Connect xz.
- Step 5: Denote $xz \cap L$ by s.

Prove that s does not depend on x, y.



Problem 3. Under the condition of Problem 2, we take $D = L - \{u\}$ and define $\oplus$ and $\otimes$ on D as follow:

- For p, q distinct from o, $p \oplus q = r$. Also, $p \oplus o = o \oplus p = p$.
- For p, q distinct from o, i, $p \otimes q = s$. Also, $p \otimes o = o \otimes p = o$ and $p \otimes i = i \otimes p = p$.

Prove that

(1) Take $\pi = \mathbb{RP}^2, L = \langle 0,1,0\rangle, M = \langle 0,0,1\rangle, o = [0,0,1], i = [1,0,1]$. What is $(D, \oplus, \otimes)$ in this case?
(2) For general π, $(D, \oplus, \otimes)$ is a division ring. We denote it by $D(\pi)$.
(3) With π fixed, choosing different L, M, o, i in π gives us isomorphic division rings.
(4) $D(\pi)$ is a field if and only if π satisfies Pappus theorem.

Problem 4. Prove that $\pi(D(\pi)) \cong \pi$ and $D(\pi(D)) \cong D$.