2023 TSINGHUA MATHCAMP RESEARCH PROJECT
ANALYSIS AND TOPOLOGY I
PROJECT B: GEOMETRY OF HYPERBOLIC PLANE

You are encouraged to read through and have an understanding of these materials, while only focus on the problems that grab your interest.

1. Basic notations

Recall that the complex number field $\mathbb{C}$ is $\mathbb{R}^2$ equipped with a multiplication

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu).$$

We usually write an element $z = (x, y)$ in $\mathbb{C}$ as $z = x + iy$, where $i = (0, 1)$, and define $\mathrm{Re}z = x$, $\mathrm{Im}z = y$, called real part and imaginary part of z, respectively. Therefore, points in the plane $\mathbb{R}^2$ can be identified with complex numbers. Then the *upper half-plane*, denoted by $\mathbb{H}$, is just the set of complex numbers with positive imaginary part.

Definition 1.1. $\mathbf{SL}(2) = \{g \in \mathbf{M}_2(\mathbb{R}) \mid \det g = 1\}$.

Definition 1.2. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2)$ and $z \in \mathbb{H}$, we define a map M_g, by $M_g(z) = \frac{az+b}{cz+d}$.

Definition 1.3. For $z \in \mathbb{C}$, $\sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}, \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}, \coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$.

2. problem list

Problem 1. We will focus on the upper half plane $\mathbb{H}$, and define a new kind of geometry on it in the following steps:

(1) We redefine *lines* on $\mathbb{H}$ as follow:
   - *Lines* (or *geodesic lines* to avoid confusion, in this case we will use *Euclid line* to denote the line in the common planar geometry) includes the Euclid lines that are perpendicular to the real axis, and the semicircle with center on the real axis. That is to say, lines on $\mathbb{H}$ are of the form

     $$l_x := \{x + it | t > 0\} \text{ for } x \in \mathbb{R}$$

     or

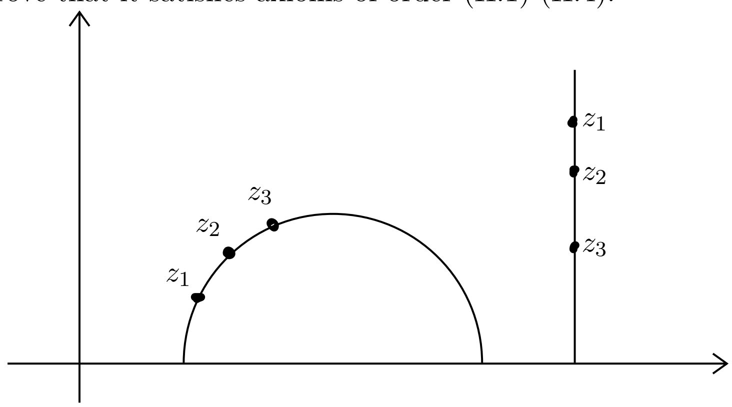
     $$l_{c,r} := \{c + re^{i\theta} | \theta \in (0, \pi)\} \text{ for } c, r \in \mathbb{R}, r > 0$$

   Prove that this new definition satisfies axioms of incidence (I.1)-(I.3).

(2) We define the *order* on lines of $\mathbb{H}$ as follow:

- We call z_2 is *between* z_1 and z_3 if z_1, z_2, z_3 are collinear, whose corresponding coordinate are t_1, t_2, t_3 or $\theta_1, \theta_2, \theta_3$(depending on which kind of line they lie in), while t_2 is between t_1 and t_3, or θ_2 is between θ_1 and θ_3.

Prove that it satisfies axioms of order (II.1)-(II.4).



(3) We redefine the *distance* between two points (or the *length of a segment*) as follow:
- For any point z_1, z_2, according to (I.1) and (I.2) we know that there exists a unique line that passes through them. Denote the corresponding coordinate as t_1, t_2 or θ_1, θ_2 (depending on which kind of line they lie in), and their distance is defined as:

$$d(z_1, z_2) = \begin{cases} |\log(\frac{t_1}{t_2})|, & \text{for } z_1, z_2 \in l_x \\ |\log(\frac{\tan\frac{\theta_1}{2}}{\tan\frac{\theta_2}{2}})|, & \text{for } z_1, z_2 \in l_{c,r} \end{cases}$$

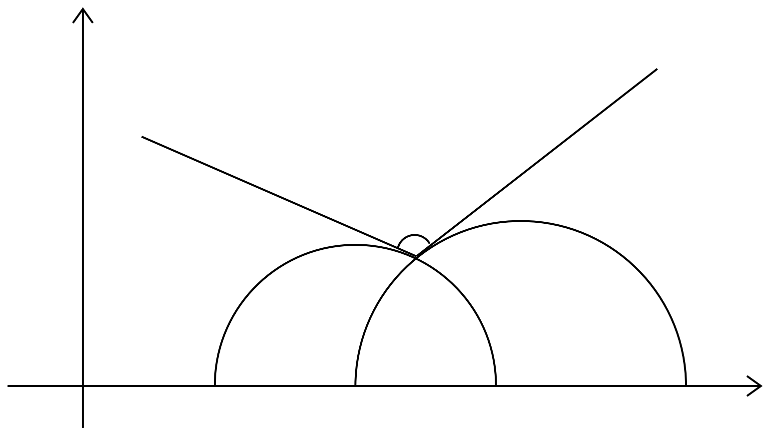
(All the logs in this problem set are natural log, that is to say, of base e.) And we define two segments are *congruent* to each other if they have the same length.

Prove that it satisfies axioms of congruence (III.1)-(III.3).

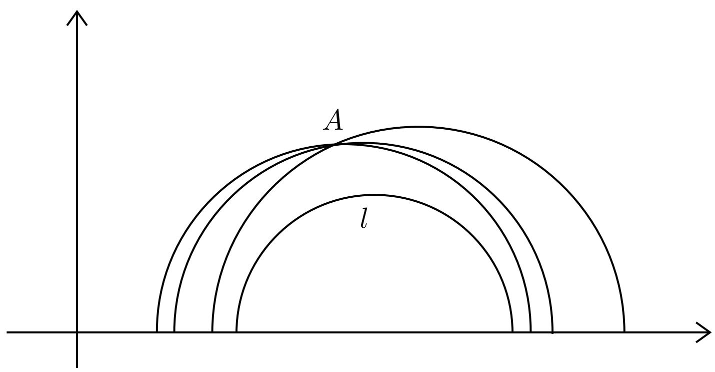
(4) Given a line l and a point A on it, we consider A decomposing l into two parts. Each such part is called a *ray* and the point A is called its endpoint. An *angle* is formed by two rays, called the sides of the angle, sharing a common endpoint, called the vertex of the angle. We measure an angle by measuring the angle formed by two Euclid rays that are tangent to its sides pointing to the same direction with the geodesic ray, and we define two angles are *congruent* to each other if they have the same measure.

(We actually omitted a lot of details here. You may want to complete it as an exercise.)

Prove that it satisfies axioms of congruence (III.4)-(III.5).



(5) For any line l and any point A not on it, show that there exist infinitely many lines pass through A and does not intersect l. Thus, Euclid's Axiom does not hold for $\mathbb{H}$.



Problem 2.

(1) Prove that the sum of interior angles of a triangle is less than π.
(2) we define the *area* of a triangle by $S_{ABC} = \pi - A - B - C$. Prove that:Area is additive. That is to say, for B, C, D collinear and D lies between B and C, we have $S_{ABC} = S_{ABD} + S_{ADC}$.

Problem 3. Prove the following trigonometry identities on $\mathbb{H}$:

(1) Hyperbolic rule of sine: $\sin A / \sinh a = \sin B / \sinh b = \sin C / \sinh c$
(2) Hyperbolic rule of cosine: $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$
(3) Dual of hyperbolic rule of cosine: $\cos C = -\cos A \cos B + \sin A \sin B \cosh c$
(4) Four-parts formula: $\cos C \cosh a = \sinh a \coth b - \sin C \cot B$

Problem 4. More about triangles on $\mathbb{H}$:

(1) For any line l and any point A, construct the (unique) line pass through A and perpendicular to l. For any two lines l_1, l_2 not intersecting, prove that there exists a unique common perpendicular.

(2) We have analogs of the concepts like perpendicular bisector, median, angle bisector or altitude in $\mathbb{H}$. Give a definition of them, and show that:
   (a) The medians of a triangle are still concurrent. The same is true for inner angle bisector and altitude as well. We still call these points centroid, incenter and orthocenter.
   (b) Unfortunately, the perpendicular bisectors of a triangle does not always concurrent.
   (c) Even more unfortunately, even if the perpendicular bisectors concurrent (the point is still called circumcenter), the orthocenter, the circumcenter, the centroid collinear (the "Euler line" still exists) if and only if the triangle is isosceles.
   (d) Half-open problem: Can you come up with some other way to define other centers of a triangle, and try to see if any of these centers collinear?

(3) The shape of a *circle* (or "geodesic circle") $C_{z_0,r} := \{z \in \mathbb{H} | d(z, z_0) = r\}$ is still an Euclid circle. Write the expressions of its Euclid center and radius as well.

(4) For any triangle, prove that we can draw a incircle centered at the incenter of a triangle and tangent to every side of it.

(5) Let r be the radius of the incircle, $s = (a + b + c)/2$, prove that $\tanh r = \sinh(s - a) \tan \frac{A}{2} = \sinh(s - b) \tan \frac{B}{2} = \sinh(s - c) \tan \frac{C}{2}$.

Problem 5. Show that

(1) for any $g \in \mathbf{SL}(2)$, M_g maps $\mathbb{H}$ to itself;

(2) $M_{g_1 g_2}(z) = M_{g_1}(M_{g_2}(z))$ for any $g_1, g_2 \in \mathbf{SL}(2)$ (in particular, $\mathrm{Id}_{\mathbb{H}} = M_g M_{g^{-1}} = M_{g^{-1}} M_g$);

(3) $M_g = \mathrm{Id}_{\mathbb{H}}$ if and only if $g = \pm I_2$. Thus, we may consider $\mathbf{PSL}(2) := \mathbf{SL}(2)/\{\pm I_2\}$ as a group acting on $\mathbb{H}$.

(4) $\mathbf{PSL}(2)$ preserves complex cross ratio $(z_1, z_2 : z_3, z_4) := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$.

(5) $\mathbf{PSL}(2)$ preserves (geodesic) lines, distance, circle, angle, and area.

(6) The transformation group of $\mathbb{H}$ preserving distance is just $\mathbf{PSL}(2)$.