Final projects for Analysis and Topology II

(draft, subject to further modifications)

Let p be a prime number. The field $\mathbb{Q}_p$ of p-adic numbers can be viewed as a non-Archimedean counterpart of the field $\mathbb{R}$. The purpose of these projects is to have a first understanding about the field $\mathbb{Q}_p$, and the p-adic analysis compared with the analysis over $\mathbb{R}$.

Background

Let $x \neq 0$ be a rational number. Write $x = p^n \cdot \frac{a}{b}$, with $a, b, n \in \mathbb{Z}$ such that $b \neq 0$ and that $p \nmid ab$. Set

$$|x|_p := p^{-n} \in \mathbb{R},$$

and call it the **p-adic absolute value** (or the **p-adic valuation**) of x. By convention, we set also $|0|_p = 0$. The following proposition is crucial in the p-adic analysis (especially the strong triangle inequality), whose proof is left to the interested readers.

Proposition 0.1. *The p-adic absolute value $|x|_p$ is well-defined. Moreover, we have*

- *for $x \in \mathbb{Q}$, $|x|_p \geq 0$, with equality if and only if $x = 0$;*
- *for $x, y \in \mathbb{Q}$, $|xy|_p = |x|_p |y|_p$;*
- *(**strong triangle inequality**) for $x, y \in \mathbb{Q}$, $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.*

In this way we deduce a metric on $\mathbb{Q}$, called the **p-adic metric**, and the induced topology on $\mathbb{Q}$ is called the **p-adic topology**. For the remaining part of this notes, unless mention explicitly the contrast, we will consider $\mathbb{Q}$ as a metric space with respect to the p-adic metric above. One checks that $\mathbb{Q}$ is not complete with respect to the p-adic metric. Let

$$\mathbb{Q}_p$$

be the completion of $(\mathbb{Q}, |\ |_p)$. By continuity, the addition and the multiplication extend to $\mathbb{Q}_p$, and $\mathbb{Q}_p$ becomes naturally a field. Moreover the p-adic absolute value on $\mathbb{Q}$ extends to a map

$$|\ |_p : \mathbb{Q}_p \longrightarrow \mathbb{R}_{\geq 0},$$

which is again referred as to the **p-adic absolute value** on $\mathbb{Q}_p$. Let

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}.$$

By the strong triangle inequality above, one checks that $\mathbb{Z}_p \subset \mathbb{Q}_p$ is a subring, called the **ring of p-adic integers**. Furthermore, we have natural inclusions

$$\mathbb{Q} \subseteq \mathbb{Q}_p, \quad \text{and} \quad \mathbb{Z} \subseteq \mathbb{Z}_p.$$

1 Project I: the topology of $\mathbb{Q}_p$

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Exercise 1.1. (1) *Show that, for any $x \in \mathbb{Q}_p$ and for any real number $r > 0$, the following sets*

$$\mathbf{B}(x,r) := \{y \in \mathbb{Q}_p \mid |y-x|_p \leq r\}, \quad \mathring{\mathbf{B}}(x,r) := \{y \in \mathbb{Q}_p \mid |y-x|_p < r\}, \quad \partial\mathbf{B}(x,r) := \mathbf{B}(x,r) \backslash \mathring{\mathbf{B}}(x,r)$$

are all open and closed.

(2) *For $n \in \mathbb{Z}$, show that $\mathbf{B}(0, p^{-n}) = p^n\mathbb{Z}_p$ and $\mathring{\mathbf{B}}(0, p^{-n}) = p^{n+1}\mathbb{Z}_p$.*

(3) *Show that, as a topological space, $\mathbb{Q}_p$ is totally disconnected: that is, the singletons $\{x\}$, for $x \in \mathbb{Q}_p$, and the empty set $\emptyset$ are the only connected subsets of $\mathbb{Q}_p$.*

(4) *Show that $\mathbb{Z}_p$ is a compact subset of $\mathbb{Q}_p$.*

Exercise 1.2. (1) *Show that, a series $\sum_{n=1}^{\infty} x_n$ of elements in $\mathbb{Q}_p$ converges if and only if $\lim_{n\to\infty} x_n = 0$.*

(2) *Show that every element $x \in \mathbb{Q}_p$ can be written in a unique way as*

$$x = \sum_{i\in\mathbb{Z}} a_i p^i, \quad a_i \in I := \{0, \dots, p-1\} \tag{1}$$

such that $a_i = 0$ for $i \ll 0$. Furthermore, $x \in \mathbb{Z}_p$ if and only if $a_i = 0$ for all $i < 0$. Compute the p-adic expansion of -1 in $\mathbb{Q}_p$.

(3) *Recall the inclusion $\mathbb{Q} \subseteq \mathbb{Q}_p$. Show that, for $x \in \mathbb{Q}_p$, $x \in \mathbb{Q}$ if and only if the coefficients a_i's in its p-adic expansion (1) is periodic, i.e., $\exists m \in \mathbb{Z}$ and $0 \neq n \in \mathbb{N}$ such that $a_i = a_{i+n}$ for every $i \geq m$.*

(4) *Let $\prod_{\mathbb{N}} I$ be a product of countably many copies of I indexed by $\mathbb{N}$. Show that the map*

$$\mathbb{Z}_p \longrightarrow \prod_{\mathbb{N}} I, \quad \sum_{i=0}^{\infty} a_i p^i \mapsto (a_0, a_1, \dots)$$

is a homeomorphism. Here we equip I with the discrete topology, and $\prod_{\mathbb{N}} I$ with the product topology.

(5) *Show that $\mathbb{Q}_p$ is not homeomorphic to $\mathbb{R}$. For different prime numbers $p \neq q$, are $\mathbb{Q}_p$ and $\mathbb{Q}_q$ homeomorphic to each other? Please justify your assertion.*

Exercise 1.3. *In this exercise, we are looking for subsets of some Euclidean space $\mathbb{R}^n$ which are homeomorphic to $\mathbb{Q}_p$ (i.e., "models" of $\mathbb{Q}_p$). For simplicity, here we merely illustrate some examples in the case $p = 2$ or 3.*

(1) *Show that the following map*

$$\mathbb{Z}_2 \longrightarrow \mathbb{R}, \quad x = \sum_{i=0}^{\infty} \frac{a_i}{2^i} \mapsto \frac{2}{3}\sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

is continuous, and defines a homeomorphism of $\mathbb{Z}_2$ onto its image. Can you recognize its image?

(2) *Consider* $\mathbb{R}^2$, $e_1 = (1,0)$ *and* $e_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. *Let*

$$\nu : \{0,1,2\} \longrightarrow \mathbb{R}^2$$

be the map given by $\nu(0) = 0$, $\nu(1) = e_1$ *and* $\nu(2) = e_2$. *Let* $b > 1$ *be a real number. Consider the map*

$$\psi : \mathbb{Z}_3 \longrightarrow \mathbb{R}^2, \quad \sum_{i=0}^{\infty} a_i 3^i \mapsto (b-1) \sum_{i=0}^{\infty} \frac{\nu(a_i)}{b^{i+1}}.$$

Show that ψ *is continuous. Moreover,*

(a) *If* $b > 2$, ψ *is injective, and gives a homeomorphism from* $\mathbb{Z}_3$ *onto its image.*

(b) *Draw a picture of* $\mathrm{im}(\psi) \subset \mathbb{R}^2$ *when* $b = 3$.

(c) *What happens if* $b = 2$*?*

(d) *Extend the construction above to a model of* $\mathbb{Q}_3$ *in* $\mathbb{R}^2$.

2 Project 2: elementary calculus over $\mathbb{Q}_p$

Exercise 2.1. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series of elements in* $\mathbb{Q}_p$.

(1) *Show that, if* $\sum_n a_n$ *converges, it converges unconditionally, i.e., for any reordering of the terms* $a_n \to a'_n$, *the series* $\sum_n a'_n$ *also converges.*

(2) *Compare (1) to what happens in the real case.*

Exercise 2.2. *In this exercise, we will show that one cannot have a reasonable ordering "*$\leq$*" as in the real case. Nevertheless, an analogous notion of "sign" can be defined for* $\mathbb{Q}_p$.

(1) *Show that there does not exist any partial order* $\leq$ *on* $\mathbb{Q}_p$ *satisfying the properties below:*

- $-1 \leq 0 \leq 1$;
- *"*$\leq$*" is compatible with the addition and the multiplication of* $\mathbb{Q}_p$ *in the evident way; and*
- *for a sequence* $\{a_n\}$ *of elements in* $\mathbb{Q}_p$ *converging to* $a \in \mathbb{Q}_p$, *if* $a_n \geq 0$ *for every* n, *then* $a \geq 0$.

(2) *For* $K = \mathbb{R}$ *or* $\mathbb{Q}_p$, *let* $K^* = K \setminus \{0\}$. *For* $x, y \in K^*$, *we denote by* $[x,y]$ *the smallest disk*[1] *containing both* x *and* y. *Define* $x \sim y$ *if* $0 \notin [x,y]$.

(a) *Show that "*$\sim$*" is an equivalence relation on* K^*.

(b) *Assume* $K = \mathbb{R}$. *Describe all the equivalence classes of* $\mathbb{R}^*$ *relative to* $\sim$. *Show that the map*

$$\mathrm{sgn} : \mathbb{R}^*/\sim \longrightarrow \{\pm 1\} \subset \mathbb{R}, \quad [x] \mapsto \frac{x}{|x|}$$

is well-defined and is bijective.

(c) *What can you say when* $K = \mathbb{Q}_p$*?*

Exercise 2.3. *A function* $f : \mathbb{Q}_p \to \mathbb{Q}_p$ *is called locally constant, if for each* $x \in \mathbb{Q}_p$, *there exists some open subset* $U \ni x$, *such that* f *is constant on* U.

[1] That is, a subset of the form $\{a \in K \mid |a - a_0| \leq r_0\}$ for some $a_0 \in K$ and $r_0 \geq 0$.

(1) *Show that locally constant functions on $\mathbb{Q}_p$ are continuous. Moreover, they are differentiable with derivation identically* 0*. Also what are the continuous locally constant real-valued function defined over* $\mathbb{R}$*?*

(2) *Show that, for every continuous function $f : \mathbb{Q}_p \to \mathbb{Q}_p$, and for any real number $\epsilon > 0$, there exists some locally constant function $g : \mathbb{Q}_p \to \mathbb{Q}_p$ so that*

$$|f(x) - g(x)| < \epsilon, \quad \forall x \in \mathbb{Q}_p.$$

If moreover the image of f is contained in some compact subset of $\mathbb{Q}_p$, show that we can even choose locally constant function g so that it has only finitely many different values.

(3) *Consider the following function*

$$f : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p, \quad \sum_{n \geq N} a_n p^n \mapsto \sum_{n \geq N} a_n p^{2n}.$$

Show that f is injective, continuous and differentiable, with derivation identically 0*.*

3 Project 3: continuous functions over $\mathbb{Z}_p$

Recall the inclusion $\mathbb{Q} \subset \mathbb{Q}_p$, which induces an inclusion $\mathbb{N} \subset \mathbb{Z}_p$. In the following, a sequence $\{x_n\}_{n \in \mathbb{N}}$ a elements in $\mathbb{Q}_p$ is often identified with the map below defined over $\mathbb{N}$:

$$f : \mathbb{N} \longrightarrow \mathbb{Q}_p, \quad n \mapsto x_n.$$

Exercise 3.1. *Show that $\mathbb{N}$ is dense in $\mathbb{Z}_p$. Deduce that, for a sequence $\{x_n\}$ of elements in $\mathbb{Q}_p$, there exists at most one continuous function $g : \mathbb{Z}_p \to \mathbb{Q}_p$ such that $g(n) = x_n$ for every $n \in \mathbb{N}$.*

If we can find such a continuous function g as above, we then say that the sequence $\{x_n\}$ can be **interpolated**.

Exercise 3.2. *Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of elements in $\mathbb{Q}_p$, with $f : \mathbb{N} \to \mathbb{Q}_p$ the corresponding map. Show that the following assertions are equivalent.*

(1) *The sequence $\{x_n\}$ can be interpolated.*

(2) *The map f is uniformly continuous. Here $\mathbb{N}$ is viewed as a subset of $\mathbb{Q}_p$.*

(3) *For any $\epsilon > 0$, there exists an integer $N > 0$, such that $n = m + p^N$ implies $|x_n - x_m|_p < \epsilon$.*

Exercise 3.3. (1) *Let $\{a_n\}$ be a nonconstant Cauchy sequence of p-adic numbers. Show that it cannot be interpolated.*

(2) *Show that, for $a \in \mathbb{Z}_p$, the sequence $1, a, a^2, \ldots$ can be interpolated if and only if $a \in 1 + p\mathbb{Z}_p$.*[2]*.*

For $n \in \mathbb{N}$, let

$$\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!}.$$

In particular, $\binom{x}{0} = 1$ by convention.

Exercise 3.4. (1) *Show that, viewed as a function on x, $\binom{x}{n}$ is uniformly continuous on $\mathbb{Z}_p$, and $\binom{x}{n} \in \mathbb{Z}_p$ for all $x \in \mathbb{Z}_p$.*

[2] This allows us to consider the continuous exponential function a^x, $x \in \mathbb{Z}_p$

(2) *Let* $\{a_n\}$ *be a sequence of elements in* $\mathbb{Q}_p$, *let*

$$F(x) := \sum_{n=0}^{\infty} a_n \binom{x}{n}.$$

Show that this series converges on $\mathbb{Z}_p$ *if and only if* $\lim_{n\to\infty} a_n = 0$.

For $\{x_n\}$ a sequence of elements in $\mathbb{Q}_p$, with $f : \mathbb{N} \to \mathbb{Q}_p$ the corresponding map, let

$$x_n^* := \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{n-k}.$$

The **interpolation series** of $\{x_n\}$, or equivalently of f, is given by the following formula

$$f^*(x) = \sum_{n=0}^{\infty} x_n^* \binom{x}{n}.$$

Exercise 3.5. (1) *Show that* $f = f^*$ *as a function defined over* $\mathbb{N}$.

(2) *If* f *admits a second representation*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \quad x \in \mathbb{N},$$

show that $a_n = x_n^*$.

Exercise 3.6 (Mahler's Theorem)**.** *Let* $f : \mathbb{N} \to \mathbb{Q}_p$ *be a uniformly continuous function. Then the interpolation series* f^* *converges uniformly to a uniformly continuous function over* $\mathbb{Z}_p$. *As a corollary, show that, every continuous function* $F : \mathbb{Z}_p \to \mathbb{Q}_p$ *can be uniformly approximated by polynomials.*[3]

[3] Compared with the real case, the way of the approximation depends naturally on the function F in the p-adic case.