Homework 0

About this homework: This is a list of basic problems to test your proficiency in elementary aspects of classical mechanics and proper use of calculus to approach dynamical systems. This homework will not carry a formal grade, but you should work on understand how to solve these problems. Some of them may be a bit advanced but all are very standard. Feel free to ask for help to your coaches.

Problem 1

Consider a (point-like) particle moving in $\mathbb{R}^2$, whose trajectory (position at a time t) as a function of time is given by a function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, where $\sigma(t) = (x(t), z(t))$ for the functions:

$$x(t) = x_0 + \frac{m}{c}v_{x,0} - \frac{m}{c}v_{x,0}e^{-\frac{ct}{m}}$$

$$z(t) = z_0 + \frac{m}{c}(v_{z,0} - gt) + \frac{m^2 g}{c^2} - \frac{m}{c}(\frac{mg}{c} + v_{z,0})e^{-\frac{ct}{m}}$$

where m is the mass of the particle, g is the gravitational acceleration (on Earth) given approximately by 9.8 in SI units, and $v_{z,0}$, $v_{x,0}$ are constants with units of velocity. c is also a constant.

- What are the units of the constant c?
- Compute the velocity of this particle at a time t.
- Compute the acceleration of this particle at a time t.
- Compute the angular momentum of this particle at a time t.
- Newton law $F = ma$ should be valid for this particle. Determine the force that is applied to this particle at every time t.
- Can you use your previous result to write a differential equation for σ? what can you learn from this expression?

Problem 2

A large class of forces in classical and quantum mechanics corresponds to what is called conservative forces. That is, these are forces F that can be written in terms of a differentiable function $V : \mathbb{R}^3 \to \mathbb{R}$ (where you should think of $\mathbb{R}^3$ as the space) by

$$F = -\nabla V = -\left(\frac{dV}{dx}, \frac{dV}{dy}, \frac{dV}{dz}\right)$$

so, the force is a vector at every point in space, formally we can write F is a function $F : \mathbb{R}^3 \to \mathbb{R}^3$ and F is independent of time.

- Consider the special case that V is only a function of $\|r\|$ where $\|\|$ denotes the Euclidean norm for any $r = (x, y, z) \in \mathbb{R}^3$. We write $V(\|r\|)$. Determine an expression for F associated to this potential. If you find this too complicated, try the following explicit expression for V:
$$V(\|r\|) = \frac{c}{\|r\|^2}$$
where c is a constant (which units?) and ignore problems of differentiability at $r = 0$ (i.e. you can consider $r \neq 0$).

- Show that the angular momentum L of a particle moving under the action of such force is conserved.

- Show that the energy E of the particle is a conserved quantity.

- There is one more conserved quantiy in this system, can you find it?. **Hint**: Try computing $\frac{d(p\times L)}{dt}$ where p is the momentum.

- Write the trajectory of a particle as $r(t) = (r_1(t), r_2(t), r_3(t))$ and its momentum as $p(t) = (p_1(t), p_2(t), p_3(t))$. Then, we define the Poisson bracket of two functions $f(p, r)$ and $g(p, r)$ as
$$\{f, g\} = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial r_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial r_i} \frac{\partial f}{\partial p_i} \right).$$
Write the angular momentum as $L = (L_1, L_2, L_3)$ and write it in terms of p and r. Do the same for the energy E. Then compute
$$\{E, L_i\}$$
for $i = 1, 2, 3$. Since we are on it, try also
$$\{p_i, L_j\}$$
for any $i, j \in \{1, 2, 3\}$.

Problem 3

We learned about the important concept of work made by a force $F : \mathbb{R}^3 \to \mathbb{R}^3$ along the trajectory $r : \mathbb{R} \to \mathbb{R}^3$ of a particle. We will see that a general expression for the work takes the form:
$$W(t_0, t_1) = \int_{t_0}^{t_1} F(r(t)) \cdot \frac{dr(t)}{dt} dt$$
Where $a \cdot b$ denotes the usual dot product of vectors in Euclidean space. Note we emphasized in the formula, by writing $W(t_0, t_1)$ that the work made by a force is defined for a path in space $\mathbb{R}^3$ (i.e., work is not an 'instantaneous' quantity like angular momentum or velocity, it only make sense to talk about work made by a force along a path).

- Compute the work done by the force of problem 1, along a trajectory between the times $0 \leq t_0 < t_1$.
- Can you write a force F (not identically 0), possibly depending on time, whose work, along any trajectory vanishes?. **Hint**: The force can depend on quantities associated to the particle.
- Compute an expression for the work done by a conservative force. Is there something special about it? if it's hard to see, consider writing it for an example.
- Is the force of problem 1 conservative?

Problem 4

Consider the quantum harmonic oscillator. This is a physical system whose properties are completely determined by the differential operator

$$\widehat{H} := -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{1}{2}m\omega^2 x^2$$

where the constant m is the mass of the particle, $\omega = \sqrt{\frac{k}{m}}$ is the frequency (k is a constant with units of $\frac{\text{force}}{\text{distance}}$) and $\hbar$ is the Planck constant, which has units of energy$\cdot$time. This sistem has states, which are eigenfunctions of $\widehat{H}$ and we can define an energy associated to each of them. It is known that the state of minimal energy has energy larger than 0. Without solving any differential equation or using any a priori knowledge of quantum mechanics, estimate the minimal energy of this system.

Problem 5

Given two points $\vec{x}_1$ and $\vec{x}_2$ (in three dimensions), consider a curve connecting them. Assuming the system is on the Earth surface and only subject to gravity which can be approximated by a constant force $\vec{F} = -ge_3$, $g \in \mathbb{R}_{>0}$ (constant), find the function of a curve which gives the shortest flight time if a particle flies along the curve.

Problem 6

A lantern moves up, perpendicular to the floor, at a constant speed $u \in \mathbb{R}_{\geq 0}$. Assume that the light coming out of the lantern forms a conical shape with apex at the position of the latern, covering a circular area on the floor. A rat moves on the floor at a speed v_0 towards the lantern, crossing diametrically the illuminated area. Consider the siuation that at time $t = 0$

the lantern is over the floor and the rat a distance D. Characterize the cone by a constant angle ϕ as shown in the figure. Compute the amount of time T the rat is illuminated by the lantern, as a function of u, v_0, D and ϕ. Analyze and interpret your result for T very large and very small.