From Classical to Quantum Mechanics

Projects

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A. Periodic potential

In this problem, we work on one dimension. Consider first a particle in a periodic potential, with period $a \in \mathbb{R}$, then the Hamiltonian operator is periodic:

$$\widehat{H}(\hat{x}+a,\hat{p}) = \widehat{H}(\hat{x},\hat{p})$$

- Show that this implies $\widehat{H}$ commutes with the translation operator $\widehat{U}_a := \exp(-ia\hat{p}/\hbar)$. How does this simplify the problem of finding the eigenfunctions of $\widehat{H}$?

- Argue that the eigenfunctions of $\widehat{U}_a$ have eigenvalues in the unitary circle. Write the eigenvalues as (this is just a convention) e^{ika} where $k \in \mathbb{R}$ is some value to be determined (in general, it depends on the details of the problem).

- Show that in the coordinate representation, the eigenfunctions of $\widehat{H}$ can be written as $\psi_k(x) = e^{ikx}u_k$ where $u_k(x+a) = u_k(x)$ is some periodic function. This is known as Bloch's theorem (or, more precisely, a simplified version of it).

- Use your previous results to solve for the eigenfunctions of the periodic potential (Kronig-Penney)

$$V(x) := \sum_{n=-\infty}^{\infty} W_0 \delta(x-na) \qquad W_0 \in \mathbb{R}$$

  where $\delta(x)$ is the Dirac delta function.

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B. Liouville's theorem

Liouville's theorem is an important property of the flow of states in the phase space of a Hamiltonian system. Assuming that the Hamiltonian is time independent, it says that this flow preserves volume.

1. Explain the theorem, and a proof.
2. Find a suitable system, or systems, which you are interested in. Use a computer to solve the equations of motion, and study the behaviour of the flow of states.
3. Write an explanation of your system, and how the theorem applies to it. Illustrate your explanation with pictures made during your computer work.

One possible reference is the following.

- V. Arnold 'Mathematical Methods of Classical Mechanics'. Part II, Chapter 3, 16.

A possible system is as follows.

- Consider a light rod of length R fixed at one end, with mass m attached at the other, swinging freely under gravity g in a vertical plane to make a pendulum. The Lagrangian is given by

  $$\mathcal{L} = \tfrac{1}{2} m R^2 \dot{\theta}^2 + mgR\cos\theta$$

  with θ the angle from the downward vertical.

C. Central potential

This problem is about the behaviour of a quantum charged particle on a constant magnetic field B (without loss of generality, you can take $B = B_0 e_2$, where B_0 is a real constant).

1. The classical Hamiltonian of this particle is given by

   $$H = \frac{1}{2m}\left(p - \frac{q}{c}A\right)^2$$

   where m is the mass of the particle, q its charge and c is the speed of light in the vacuum. p is the momentum of the particle and A is the vector potential for B, i.e. $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector function defined by

   $$B = \nabla \times A.$$

Show that H makes sense, by computing the Hamilton equations of motions and showing that they reduce to the equation of motion of a classical charged particle on a constant magnetic field, that is

$$\frac{d^2x}{dt^2} = \frac{q}{mc} v \times B, \qquad v := p - \frac{q}{c}A.$$

2. It is clear that $B = \nabla \times A$ does not uniquely define A. Write at least three different vector functions A (in Cartesian coordinates) that give the same $B = B_0 e_3$. This freedom is called gauge symmetry.

3. We will now move on to the quantization of this problem. Consider the Hamiltonian operator (in the coordinate representation)

$$\widehat{H} = \frac{1}{2m}\left(\hat{p} - \frac{q}{c}A\right)^2$$

where $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$ satisfy the canonical commutation relations. Define the operators $\Pi_i = \hat{p}_i - \frac{q}{c}A_i$ for $i = 1, 2$ and compute their commutation relations.

4. Now, set $\hat{p}_3 = 0$ and $A_3 = 0$ (why does the latter makes sense in general?), and show that, by defining appropriate linear combinations of Π_1 and Π_2, the Hamiltonian operator $\widehat{H}$ takes the form of an harmonic oscillator. Compute the energy levels and explain how they qualitatively depend on B. These are known as the Landau levels.

5. Note that, in the previous question we did not use any particular gauge (i.e. a particular choice for a solution of $B = \nabla \times A$). So, the energy levels do not depend on the choice of gauge. Argue that the eigenfunctions do indeed depend on the choice of gauge. Then, solve for the ground state in the gauge $A = B_0 x e_2$. **Hint**: Try a solution of the form $\psi(x, y) = f(x)g(y)$.

6. Consider now the gauge $A = -\frac{1}{2}B_0 y e_1 + \frac{1}{2}B_0 x e_2$ and solve for the ground state on it. **Hint**: It may be convenient to work using the complex coordinates $z = x + iy$ and $\bar{z} = x - iy$.

D. Symplectic geometry and conserved quantities

Symplectic manifolds are a class of manifolds which appear naturally when we study mechanics from a mathematical point of view. In this problem we will work in the simpler setting of symplectic vector spaces

1. Define of a symplectic vector space (a good reference can be Ch. 1 of the book 'Lectures of symplectic geometry' by A. Cannas da Silva). In addition define the concept of Hamiltonian vector field in a symplectic vector space.

2. Apply your previous result to the case that the symplectic vector space is the phase space of a system of a single point particle, moving on $\mathbb{R}^3$ under the action of a central force sourced at the origin i.e. the potential takes the form $U = U(\|\vec{x}\|)$. Describe the symplectic form on these coordinates. What is the meaning of the Hamiltonian vector field in this space?

3. Consider the case $U = U(\|\vec{x}\|)$ is the gravitational potential and show that the angular momentum, the energy, and the Laplace-Runge-Lenz vector are conserved quantities (you may have already done some of these in Hw 0).

4. **Bonus**: From Noether theorem, we know that each conserved quantity is associated to a symmetry. What is the symmetry associated to the Laplace-Runge-Lenz vector? One way to proceed is suggested: you can assume any conserved quantity (for example a component of the Laplace-Runge-Lenz vector) can be used to construct a Hamiltonian vector, then argue, by analogy to the time translation case, that the infinitesimal version of these symmetries must corresponds to the flow along the vector. After obtaining an explicit version for these symmetries, apply them to the Lagrangian and check they are indeed symmetries of the action.

Helpful references include the following.

- V. Arnold 'Mathematical Methods of Classical Mechanics'. Part II, Chapter 4 and Part III, Chapter 8. .

E. Reference Systems

Consider the classical Newton's system in the space $\mathbb{R}^2$. Newton believed that the space and time are independent aspect of objective reality. Following Newton, we separate the space $\mathbb{R}^2$ and time $\mathbb{R}$ and believe that they has no direct relations. When we try to say this **absolute space-time**, we will use the notation $\mathbb{R}^{(2,1)} := \mathbb{R}^2 \oplus \mathbb{R}$.

1. Give a mathematical definition of the inertial frames of $\mathbb{R}^{(2,1)}$.

2. Consider a translation on $\mathbb{R}^{(2,1)}$, this will induce a transformation between two different inertial frames. Does this transformation preserve Newton's Law? Prove your statement.

3. How about a rotation on $\mathbb{R}^2$?

4. Suppose two inertial frames have different velocity v_1 and v_2, how are these two inertial frames related? Find a transformation to describe this relation. Such transformation is called the **Galilean boost**. Does this transformation preserve Newton's Law? Prove your statement.

5. Define Galilean transformations on $\mathbb{R}^{(2,1)}$. And prove that they form a group.

From now on, we denote the Galilei group on $\mathbb{R}^{(2,1)}$ by $\mathrm{Gal}(2,1)$.

Based on this fact, one can ask several questions in many different ways. You have to choose ONE of the following problems as your research direction. And do NOT study more than one direction at the same time.

1. Geometry associated to Galilei group

   (a) The Galilei group $\mathrm{Gal}(2,1)$ has a subgroup that is generated by the rotation on $\mathbb{R}^2$ and the translation on $\mathbb{R}^2$ (not on $\mathbb{R}^{(2,1)}$), denoted by E^2. Describe E^2 and its action on $\mathbb{R}^2$.

   (b) Find the definition of a metric space. Then define a metric on $\mathbb{R}^2$ by your intuition in reality. Prove that the E^2-action preserves metric on $\mathbb{R}^2$. Therefore, the metric is an **invariant** on $\mathbb{R}^2$ under the action of E^2.

   (c) Indeed, we can prove that the group which preserves the metric on $\mathbb{R}^2$ can only be E^2. So $\mathrm{Gal}(2,1)$ does not preserve the metric on $\mathbb{R}^2$. Prove it.

   (d) (*) Extend the metric on $\mathbb{R}^2$ to a function on $\mathbb{R}^{(2,1)}$. So this function is also not an invariant of $\mathbb{R}^{(2,1)}$ under the $\mathrm{Gal}(2,1)$-action. Try to find the invariants of $\mathbb{R}^{(2,1)}$ under the $\mathrm{Gal}(2,1)$-action and explain the physical significant. (Hint: Maybe you can first study the case in $\mathbb{R}^{(1,1)}$. Or maybe you can consider when the metric on $\mathbb{R}^2$ could be an invariant.)

   (e) (**) Based on the invariants you just found, prove that the group which preserves the invariants can only be $\mathrm{Gal}(2,1)$.

2. Special Relativity in space-time $\mathbb{R}^{2,1}$.

(a) Find some evidence that some physical rules and quantities are not covariant under the action of the Galilei group. And find some evidence that the speed of light is invariant in different reference systems.

(b) (*) Based on these two facts (you have to admit them!), try to derive the transformation between two inertial frames whose y-component velocity are the same, but x-component velocity differs by v. Such transformations are called **boosts**.

(c) Show that when $v \ll c$ (or equivalently, $c \to \infty$), the boosts degenerate to the Galilean boosts.

(d) Show that the boosts in two directions, together with rotations on $\mathbb{R}^2$, form a group, denoted by $\mathrm{SO}(2,1)$. We will call this group the Lorentz group.

(e) Show that the translation on $\mathbb{R}^{2,1}$ together with $\mathrm{SO}(2,1)$ form a group, denoted by $\mathrm{Poin}(2,1)$. And describe the group action on $\mathbb{R}^{2,1}$

(f) Let $p_1 := (x_1, y_1, t_1)$ and $p_2 := (y_1, y_2, t_2)$ be two points of $\mathbb{R}^{2,1}$, we define a distance of these two points as follows:

$$d^2(p_1, p_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 - c^2(t_1 - t_2)^2$$

Show that the Poincare group $\mathrm{Poin}(2,1)$ preserves this distance.

F. Topological Orders

In this project, you will study a quantum many-body system which is very simple but interesting, the toric code model. The 2d toric code model on a square lattice is defined as follows. There is a spin-1/2 on each edge (or link) of the lattice. In other words, the local degree of freedom $\mathcal{H}_i$ on each edge i is a two-dimensional Hilbert space $\mathbb{C}^2$. The total Hilbert space is $\mathcal{H}_{tot} := \bigotimes_i \mathcal{H}_i = \bigotimes_i \mathbb{C}^2$.

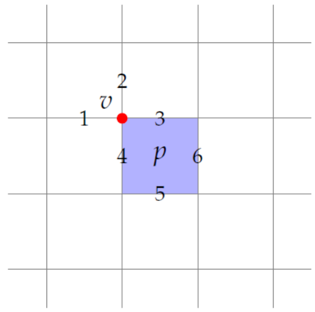

Figure 6: the toric code model

Let us recall Pauli matrices acting on $\mathbb{C}^2$:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. Show that $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathrm{id}$, where $i, j = x, y, z$, $\delta_{ij} = 1$ only if $i = j$, other $= 0$ and id is the identity matrix. As a consequence, $\sigma_i^2 = 2\mathrm{id}$ for $i = x, y, z$.

For each vertex v and plaquette p we define a vertex operator $A_v := \prod_i \sigma_x^i$ and a plaquette operator $B_p := \prod_j \sigma_z^j$ acting on adjacent edges. Here $\sigma_x^i = \cdots \otimes \mathrm{id} \otimes \sigma_x \otimes \mathrm{id} \otimes \cdots$ is the operator that acts on H_i as σ_x and acts on other local Hilbert spaces as identities. For example, the operators in Figure 6 are

$$A_v = \sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4, \quad B_p = \sigma_z^3\sigma_z^4\sigma_z^5\sigma_z^6$$

2. Show that $[A_v, B_p] = 0$ for any vertex v and any plaqutte p.

The Hamiltonian of toric code model is defined to be:

$$H := \sum_v (1 - A_v) + \sum_p (1 - B_p)$$

where the summation takes over all vertices v and all plaquettes p.

3. Show that $[H, B_p] = 0 = [H, A_v]$ for any vertex v and any plaqutte p.

4. Find the ground state of this Hamiltonian and find the corresponding energy. What are eigenvalues of the operators A_v and B_p on ground state?

It follows that the total Hilbert space can be decomposed as the direct sum of common eigenspaces of all A_v and B_p operators.

5. Now supposed this lattice model is defined on a genus g closed surafce. Find the ground state degeneracy i.e. the dimension of the ground state subspace by Euler's formula

$$V - E + F = 2 - 2g$$

   where V is the number of the vertices, E is the number of edges, F is the number of faces.

What you just find is a topological invariant, since it only depends on the genus g of the surface, which means it is invariant under any isomorphism between topological spaces. Such quantum many-body system is called a *topological order*.

A quantum many-body system is "topological" intuitively implies that it is "invariant" under some small perturbations. Such perturbations are characterized by so-called *local operators* i.e. operators are defined in a bounded region and act on the local Hilbert space in the bounded region. In toric code model, for example, σ_x^i is a local operator since it only acts on the i-edge.

There are also some *topological excitations* (or *topological defects*) invariant under the action of local operators. More explicitly, a *topological defect* is a subspace of the total Hilbert space that is invariant under the action of local operators. For example, the ground state belongs to the trivial topological excitations, which we denote **1**.

6. (*) Consider the following state $|\psi_{v_0}\rangle$ at the vertex v_0 satisfying

$$\begin{cases} A_{v_0}|\psi_{v_0}\rangle = -|\psi_{v_0}\rangle, \\ A_v|\psi_{v_0}\rangle = |\psi_{v_0}\rangle \text{ for all vertices } v \neq v_0, \\ B_p|\psi_{v_0}\rangle = |\psi_{v_0}\rangle \text{ for all plaquttes } p \end{cases} \tag{1}$$

   Show that it cannot be annihilate by local operators σ_z^i where i is adjcent to vertex v_0. Thus $|\psi_{v_0}\rangle$ belongs to a non-trivial topological excitation which we denote e.

7. Show that the states $|\psi_{v_1}\rangle$ and $|\psi_{v_2}\rangle$ for any vertices v_1 and v_2 belongs to the same topological excitation e.

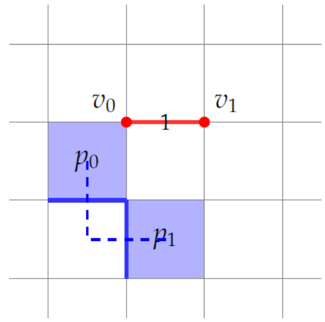

Figure 9: topological excitations of toric code

Now we can explain why these topological excitations are "topological": the local operators only changes the position of the topological excitations $|\psi_{v_0}\rangle$, and would not change a topological excitation to another one. Geometrically, it is equivalent to deform the background a little bit such that the excitations moves to the new position.

Similarly, for each plaqutte p_0 there is a state $|\phi_{p_0}\rangle$ satisfying the following property:

$$\begin{cases} B_{p_0}|\phi_{p_0}\rangle = -|\phi_{p_0}\rangle, \\ B_p|\phi_{p_0}\rangle = |\phi_{p_0}\rangle \text{ for all plaquttes } p \neq p_0, \\ A_v|\phi_{p_0}\rangle = |\phi_{p_0}\rangle \text{ for all vertices } v \end{cases} \tag{2}$$

It is not hard to see that $|\phi_{p_0}\rangle$ belongs to another topological excitation which we denote m.

8. Notice that the local operators σ_z^i will move the topological excitation e on the lattice. Consider the following diagram,

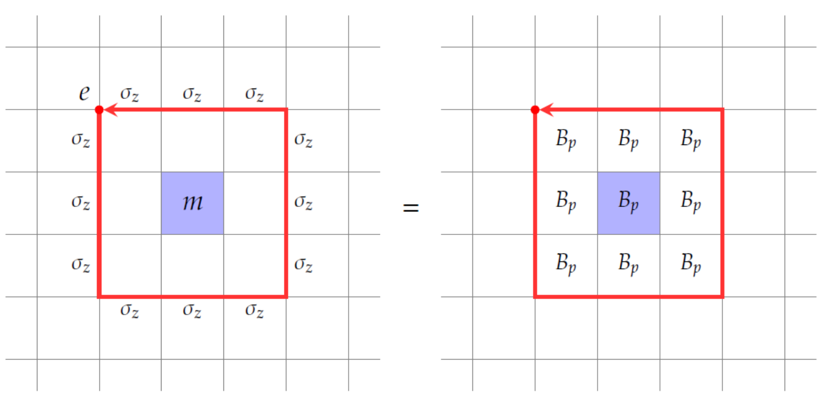

Figure 33: the double braiding of e and m in the toric code model

we can use local operators to move e around m. This is called the *double braiding* of e and m. Compute it, you would get a number.

9. Show that we can always enlarge the loop by some local operators without changing the number of double braiding. So the double braiding is a non-local property of topological excitations. (Hint: you may first consider the action of a local operator by twice.)

G. Fields and Differential Forms

1. The *Coulomb's law* states that the force between two particles with charge q, q' has the form

$$\mathbf{F} = \frac{qq'\mathbf{r}}{4\pi\varepsilon_0 r^3},$$

where $\varepsilon_0 = 8.8541878 \times 10^{-12}$ F/m is the *vacuum permittivity* and r is the distance. The *electric field* $\mathbf{E}$ is defined by the ratio of the force experienced by a charged particle to its charge q, i.e. $\mathbf{E} = F/q$. Show that $\nabla \cdot \mathbf{E} = \rho/\varepsilon$ and $\nabla \times \mathbf{E} = 0$, where ρ is the *charge density*.

2. The *Biot-Savart law* states that the steady *electric current* $\mathbf{j}$ excite the *magnetic field* $\mathbf{B}$ via

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{j}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}'.$$

where $\mu_0 = 4\pi \times 10^{-7}$ H/m is the *vacuum permeability*. Show that $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$.

3. Show that the current $\mathbf{j}$ and the charge density ρ satisfy the *continuity equation*

$$\nabla \cdot \mathbf{j} + \frac{d\rho}{dt} = 0.$$

4. The *Farady' law* states that the changes of the *magnetic flux* in a fixed circuit l generate the *electromotive force* $\mathcal{E}$ then induce the electric field E.

$$\oint_l \mathbf{E} \cdot d\mathbf{l} = \mathcal{E} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}.$$

Show that $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.

5. We have the equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, taking the divergence on the both sides we get

$$0 = \nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{j}.$$

The result contradicts to the continuity equation. Hence Maxwell rewrite the equation

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}).$$

Show that the divergence of the right hand side is 0.

6. The above results give the *Maxwell's equations*

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho/\varepsilon_0; \\ \nabla \cdot \mathbf{B} = 0; \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \\ \nabla \times \mathbf{B} = \mu_0(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}). \end{cases}$$

Show that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ both satify the wave equation

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$

Calculate the speed of the wave v and explain this result.

Consider the space $\mathbb{R}^n = (x^1, \ldots, x^n)$, here the i in dx^i just an index rather than the power. Define the *wedge product* of the differrentials $dx^1, \ldots, dx^n$, denoted by $\wedge$, which is:

(1) bi-linear: $(adx^i + bdx^j) \wedge dx^k = adx^i \wedge dx^k + bdx^j \wedge dx^k$,

$$dx^i \wedge (adx^j + bdx^k) = adx^i \wedge dx^j + bdx^i \wedge dx^k;$$

(2) anti-commutative: $dx^i \wedge dx^j = -dx^i \wedge dx^j$;

(3) associative: $(dx^i \wedge dx^j) \wedge dx^k = dx^i \wedge (dx^j \wedge dx^k)$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, called a 0-form on $\mathbb{R}^n$. Take the differential of f:

$$df = \frac{\partial f}{\partial x^1}dx^1 + \cdots + \frac{\partial f}{\partial x^n}dx^n.$$

Then we say df is a 1-form on $\mathbb{R}^n$. Generally, a 1-form ω on $\mathbb{R}^n$ has the expression

$$\omega = \omega_i dx^i$$

where $\omega_i : \mathbb{R}^n \to \mathbb{R}$ are smooth functions on $\mathbb{R}^n$ and $dx^1, \ldots, dx^n$ form a basis of all 1-forms on $\mathbb{R}^n$.

Reminder: You should always remember the *Einstein summation convention*: sum all the repeated index, called the *dumb index*, over all possible values. Here $\omega = \omega_i dx^i := \sum_{i=1}^n \omega_i dx^i$.

For example, the wedge product of $\omega = \omega_i dx^i$ and $\theta = \theta_j dx^j$ is

$$\omega \wedge \theta = \sum_{i,j}(\omega_i dx^i) \wedge (\theta_j dx^j) = \sum_{i,j}(\omega_i \theta_j) dx^i \wedge dx^j = \sum_{i<j}(\omega_i\theta_j - \omega_j\theta_i) dx^i \wedge dx^j.$$

Then we get a 2-form $\omega \wedge \theta$. We can construct a k-form by taking the wedge product of k 1-forms. Let $\Omega^k(\mathbb{R}^n)$ denote the set of k-forms on $\mathbb{R}^n$. Then we can generalize the wedge product by

$$\wedge : \Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n) \to \Omega^{k+l}(\mathbb{R}^n).$$

7. Show that $\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} \mid i_1 < \cdots < i_k\}$ is a basis of $\Omega^k(\mathbb{R}^n)$ and calculate $\dim \Omega^k(\mathbb{R}^n)$.

8. We extend the differential operator $d : \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ to the *exterior derivative*

$$d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n), \quad \omega \mapsto d\omega.$$

   For $\omega = \omega_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where $\omega_{i_1\cdots i_k}$ is the functions of $x^{i_1}, \ldots, x^{i_k}$, $d\omega$ is given by

$$d\omega = (d\omega_{i_1\cdots i_k}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \frac{\partial \omega_{i_1\cdots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

   Show that $d^2 = d \circ d = 0$.

Define the *Hodge star operator* on the Minkowski space $\mathbb{R}^{3,1} = (x, y, z, t)$ with the Minkowski metric $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ by

$$* : \Omega^p(\mathbb{R}^{3,1}) \to \Omega^{4-p}(\mathbb{R}^{3,1})$$

Explicitly, we have the formulas

$$*1 = cdt \wedge dx \wedge dy \wedge dz, \quad *(cdt \wedge dx \wedge dy \wedge dz) = 1.$$

$$*cdt = dx \wedge dy \wedge dz, \quad *dx = cdt \wedge dy \wedge dz, \quad *dy = cdt \wedge dz \wedge dx, \quad *dz = cdt \wedge dx \wedge dy.$$

$$*(cdt \wedge dx) = -dy \wedge dz, \quad *(cdt \wedge dy) = -dz \wedge dx, \quad *(cdt \wedge dz) = -dx \wedge dy.$$

$$*(dx \wedge dy) = cdt \wedge dz, \quad *(dy \wedge dz) = cdt \wedge dx, \quad *(dz \wedge dx) = cdt \wedge dy.$$

$$*(dx \wedge dy \wedge dz) = cdt, \quad *(cdt \wedge dx \wedge dy) = dz, \quad *(cdt \wedge dy \wedge dz) = dx, \quad *(cdt \wedge dz \wedge dx) = dy.$$

9. Define the current 1-form J on $\mathbb{R}^{3,1}$ by the charge density ρ and the current $\mathbf{j} = (j_x, j_y, j_z)$

$$J = \frac{\rho}{\varepsilon_0} dt - \mu_0 j_x dx - \mu_0 j_y dx - \mu_0 j_z dz.$$

   Show that the continuity equation is equivalent to $d(*J) = 0$.

10. The electric field $\mathbf{E} = (E_x, E_y, E_z)$ can be identified with a 1-form E on $\mathbb{R}^{3,1}$

$$E = E_x dx + E_y dy + E_z dz.$$

The magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ can be identified with a 2-form B on $\mathbb{R}^{3,1}$

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.$$

The *electromagnetic field strength* 2-form F on $\mathbb{R}^{3,1}$ is defined by

$$F = E \wedge dt + B.$$

Define the adjoint operator d^* of d on $\mathbb{R}^{3,1}$ by

$$d^* = *d* : \Omega^p(\mathbb{R}^{3,1}) \to \Omega^{p-1}(\mathbb{R}^{3,1}).$$

Show that the Maxwell's equations are equivalent to $\begin{cases} dF = 0; \\ d^*F = J. \end{cases}$

H. Collective phenomena: diffusion equation

In this problem we see how (classical) fields can arise from the dynamics of multi-particle systems. Consider a system of N non-interacting particles moving in the real line $\mathbb{R}$. The position of a particle at time t is denoted $x_i(t)$, $i = 1, \ldots, N$ and satisfy the current properties:

- Every τ seconds the particle can move to the left or to the right, with probability $\frac{1}{2}$ for each.
- Every time it moves it does it by an amount $\delta \in \mathbb{R}_{>0}$, so $x_i(t + \delta) = x_i(t) \pm \delta$.

Assume then we have a 'cloud' of particles i.e. $N \gg 1$ (possibly several orders of magnitude). Then it makes sense to define a function $P(t, x)$, $P : \mathbb{R}^2 \to \mathbb{R}$, where $P(t, x)$ denotes the number of particles at position $x \in \mathbb{R}$, at time t. So, in principle x in the argument of P can only take discrete values but we will approximate it, later on, to the whole real line.

1. Explain why in principle x in the argument of P can only take discrete values and under which circumstances it makes sense to assume it is continuous.

2. Suppose we have L particles at a position x. Then we can assume, as a good approximation, that half of them will move to the right and half to the left after τ seconds. Denote then $J(t, x)$ the number of particles passing though a point between x and $x + \delta$ per time, during an interval of τ seconds i.e. J has units of time^{-1}. Find an expression for $J(t, x)$ in terms of the function P and τ.

3. Define the density of particles by $\rho(t,x) := \frac{P(t,x)}{\delta}$ and write J in terms of ρ. Then, take the limit $\delta \to 0$ and express J in this limit (you can assume $\frac{\tau}{\delta}$ remains constant in this limit). You should find that J is proportional to a partial derivative of ρ.

4. Using a similar reasoning than in 2, argue that

$$P(t+\tau, x) - P(t,x) = (J(t, x-\delta/2) - J(t, x+\delta/2))\tau$$

5. Finally, taking the limit $\tau \to 0$ (keeping $\frac{\tau}{\delta}$ constant), show that ρ satisfies the PDE:

$$\frac{\partial \rho(t,x)}{\partial t} = D\frac{\partial^2 \rho(t,x)}{\partial x^2}$$

what is the valued of the constant D?

6. In order to understand the physical meaning behind this equation, known as the diffusion equation, we will compute some exact solution for it. For this purpose, consider a solution of the form

$$\rho(t,x) = \int_{-\infty}^{\infty} \tilde{\rho}(k) e^{ikx-\omega t} dk$$

where $\tilde{\rho} : \mathbb{R} \to \mathbb{C}$ is a complex function and ω is a real constant. Moreover, we assume all the functions are integrable, so, is ok to exchange partial derivatives and integral. Show that the $\rho(t,x)$ above is a solution if

$$D\omega = k^2$$

7. Now, we give initial conditions to the system. consider

$$\rho(0,x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

where σ is a positive constant. Use this initial condition and the following expression for the Dirac delta function:

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx}$$

to find an explicit expression for $\tilde{\rho}(k)$, in this case.

8. Performing the integrals, find an explicit expression for $\rho(t,x)$. You should find a real function. Plot it, at various times and explain your results.

I. Chaotic systems

Chaotic systems are the most common systems in nature. In this project we explore some of their properties.

1. Consider a one dimensional system with the following Hamiltonian

$$H = \frac{1}{2}p^2 - K\cos x \sum_{n=-\infty}^{\infty} \delta(t/T - n)$$

where $K, T \in \mathbb{R}_{>0}$ are constants and $\delta(t)$ is the Dirac delta function. Argue that the evolution of the system can be approximated by a discrete map, as t evolves from $t = 0$, to $t = T, 2T, 3T, \ldots$ as

$$p_n = p_{n-1} - K\cos(x_{n-1}), \qquad \theta_n = x_{n-1} + p_n$$

where $p_n = p(nT)$ and $x_n = x(nT)$.

2. With the help of a software of your choice, plot different trajectories (in the phase space (x, p)) for different initial conditions for (x, p) and for different values of K. Compare with the dynamical system

$$\frac{dp}{dt} = -\frac{K}{T}\sin\theta, \qquad \frac{dx}{dt} = \frac{1}{T}p$$

when is this a good approximation? (or when it becomes worse and worse). Compare also with $K = 0$, what happens with the periodic orbits?

3. Consider now a one dimensional system given by a particle in a double well potential:

$$U(x) = R\frac{x_0^2}{8}\left(1 - \left(\frac{x}{x_0}\right)^2\right)^2$$

where $R, x_0 \in \mathbb{R}_{>0}$ are constants. With the help of a plot of $U(x)$, sketch the phase diagram for this system, with trajectories at different energies. In particular, sketch the separatrix trajectory and its energy (you don't need to find an analytic expression for this trajectory).

4. Now consider a time dependent perturbation of $U(x)$:

$$U(x, t) = U(x) - F_0 \sin(\omega t)x$$

where $\omega, F_0 \in \mathbb{R}_{>0}$ are constants. Even though conservation of energy do not longer holds after the perturbation, assume it does and for a small F_0, find an approximation for the new location of the separatrix energy (your answer must depend on t). Plot, with a software, different solutions of the equations of motion, in the phase space (you can assume the kinetic energy is $\frac{1}{2}\dot{x}^2$), for different values of F_0 (and fix the other constants). By comparing to the phase diagram for $F_0 = 0$, show, by plotting, that as F_0 increases, the trajectories around the separatrix becomes more and more complex, but away of it, they remain regular for small F_0. This is an example of how a perturbation to an integrable system transforms it into a chaotic one.

5. **Bonus**: Consider a solution $x_0(t)$ of the perturbed equation of motion and a close-by solution $x_0(t) + \delta x(t)$ where $\delta x(t)$ is a small perturbation. Linearize the equations of motion and find an differential equation for $\delta x(t)$. Integrating in numerically, show that $\sqrt{(\delta x(t))^2 + (\delta \dot{x}(t))^2}$ diverges as time increases (plot it for different values of F_0).