ACI Homework Due 8/1 in class

July 25, 2024

Problem 1. *1. Let $P = (5, 4)$ and $Q = (-1, 5)$. Find a normal vector and linear equation for the line passing through P and Q.*

2. Prove that there exists a unique line passing through two distinct points on $\mathbb{R}^2$.

3. Prove that if two lines in $\mathbb{R}^2$ have more than two intersection points, then they are the same line.

Problem 2. *Find the projection of vector* $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$ *onto vector* $\vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Problem 3. *We say a set $A \subset \mathbb{R}^2$ is bounded if A is contained in a disc with finite radius.*

Let $\mathcal{L} = \{l_1, \cdots, l_5\}$ be a line arrangement consisting of five distinct lines in $\mathbb{R}^2$. Let N_1 be the number of regions associated with $\mathcal{L}$, and N_2 the number of bounded regions. What are the possible pairs of (N_1, N_2)?

Problem 4. *1. Let $\mathcal{L} = \{l_1, \cdots, l_n\}$ be a line arrangement consisting of n distinct lines in $\mathbb{R}^2$. Find the maximal value for N_1, the number of regions associated with $\mathcal{L}$, and the maximal value for N_2, the number of bounded regions.*

2. When each of the numbers N_1 or N_2 achieves the maximal value in the last question, is it true that $\mathcal{L}$ is in general position?

3. Let $\mathcal{H} = \{H_1, \cdots, H_n\}$ be a plane arrangement in $\mathbb{R}^3$. Find the maximal value for N_1, the number of regions associated with $\mathcal{H}$, and the maximal value for N_2, the number of bounded regions.

4. When each of the numbers N_1 or N_2 achieves the maximal value in the last question, is it true that $\mathcal{H}$ is in "general position"? (How to define a plane arrangement is in general position.)

Problem 5. *1. Guess what are the all possible convex subsets in $\mathbb{R}$.*

2. Try to write down examples of sets $A \subset \mathbb{R}^2$ such that both A and its complement $\mathbb{R}^2 \backslash A$ are both convex subsets of $\mathbb{R}^2$.

3. *Guess what are the possible sets A in the previous question.*

Problem 6. *Define the cross product of vectors in $\mathbb{R}^3$ by the following algebraic formula. If*

$$\vec{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \vec{w} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

Then the cross product $\vec{v} \times \vec{w}$ is a vector in $\mathbb{R}^3$ with coordinates

$$\vec{v} \times \vec{w} = \begin{pmatrix} a_2 b_3 - b_2 a_3 \\ -a_1 b_3 + b_1 a_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

Prove that

1. $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.
2. $\vec{v} \times \vec{w}$ *is perpendicular to both $\vec{v}$ and $\vec{w}$.*
3. *If $\vec{v}$ and $\vec{w}$ has angle θ, then $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\theta)$.*
4. *Find a normal vector of the plane passing through $P = (1, 2, 3)$, $Q = (4, 5, 6)$ and $R = (1, 1, 2)$.*

Problem 7. *Find whether there exists seven lines on $\mathbb{R}^2$ such that they have the following intersection relation, see picture at the next page. The circle in the middle means a line passing through points P, Q, R. The seven points A, B, C, D, P, Q, R are all the intersection points formed by these seven lines.*

Problem 8. *Find whether there exists nine lines on $\mathbb{R}^2$ with the following intersection relations, see the picture at the last page. There are 14 intersection points $P_1, \cdots, P_{14}$. (Hint: Pappus's hexagon theorem)*

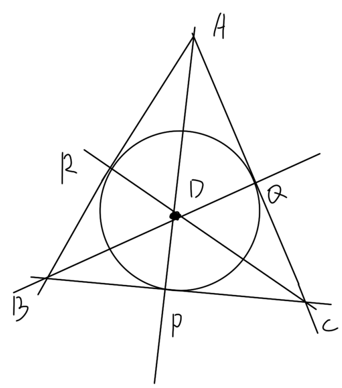

Figure 1: Picture for Problem 7

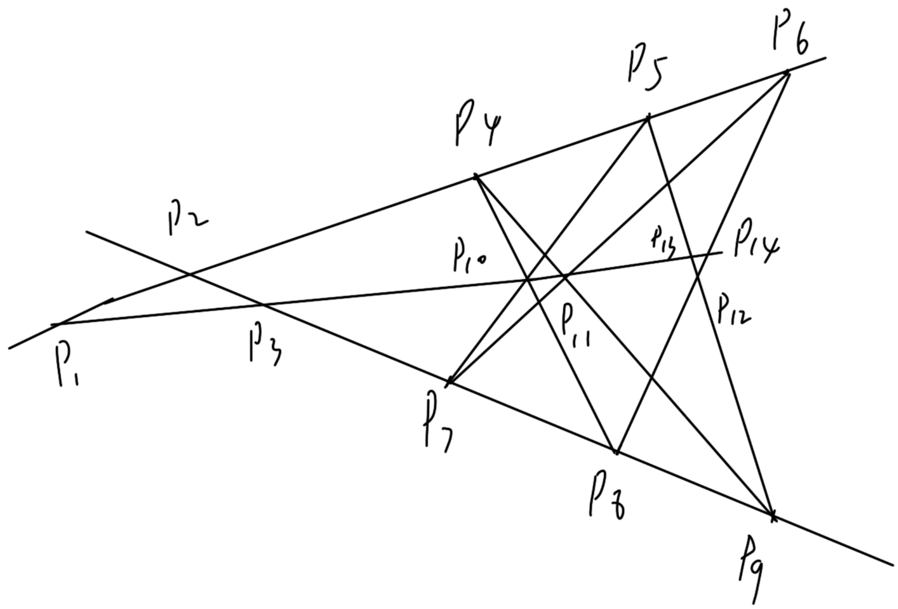

Figure 2: Picture for Problem 8