ACI Projects

August 1, 2024

Project 1. *In this project, you will compute invariants related to certain hyperplane arrangements.*

1. *Let Ω be a convex set in $\mathbb{R}^d$ defined by*

$$x_1 + x_2 + \cdots + x_d \leq 1, \text{ and } x_1 \geq 0, x_2 \geq 0, \cdots, x_d \geq 0.$$

Find all the reflections that preserve Ω and the corresponding mirror hyperplanes.

2. *Let $\mathcal{H} = \{H_1, \cdots, H_n\}$ be the hyperplane arrangement formed by the reflection mirrors. Count the number of regions cut out by this arrangement.*

3. *Find the characteristic polynomial of this arrangement.*

4. *Next, we consider the translations of $\mathcal{H}$. Let $\mathcal{H}_m$ be the set of hyperplanes that are parallel to some $H_i \in \mathcal{H}$ and have distance $D \in \{0, 1, 2, \cdots, m\}$ with H_i. Find the characteristic polynomials for $\mathcal{H}_m$ and the number of regions cut out by $\mathcal{H}_m$.*

5. *Let G be the group generated by all reflections that preserve Ω. What is the minimal number of reflections to generate G. How to characterize the minimal generating set of reflections for G?*

Project 2. *In project, you will find a counting interpretation for the coefficients of characteristic polynomial of an arrangement.*

Definition 1 (Closure)**.** *Let $\mathcal{H} = \{H_1, \cdots, H_n\}$ be a hyperplane arrangement in $\mathbb{R}^d$. Assume H_i is defined by the linear equation $f_i = 0$. Let $\Omega_1, \cdots, \Omega_n$ be the regions cut out by $\mathcal{H}$. The closure $\overline{\Omega}_j$ of Ω_j is a subset of $\mathbb{R}^d$ by adding equality for each inequality defining Ω_j. For example, if Ω_1 is defined by inequalities, $f_1 > 0, \cdots, f_n > 0$, then $\overline{\Omega}_1$ is defined by $f_1 \geq 0, \cdots, f_n \geq 0$.*

Definition 2. *For any point $u \in \mathbb{R}^d$, let $v \in \overline{\Omega}_j$ be the closest point from $\overline{\Omega}_j$ to u. In other words, the distance between v and u is equal to the distance between $\overline{\Omega}_j$ and u. If the point v is lying on a k-dimensional element in the intersection poset $P(\mathcal{H})$, and not on any $k-1$-dimensional element, then we say it has projection dimension $\dim(u, \Omega_j) = k$ from u to Ω_j.*

1. *Let $\mathcal{L} = \{l_1, l_2\}$ be a line arrangement on $\mathbb{R}^2$ defined by $l_1 \colon x_1 = 0$, and $l_2 \colon x_1 + x_2 = 0$. Find the projection dimension for any point $u \in \mathbb{R}^2$ and each region cut out by $\mathcal{L}$.*
2. *Let $\mathcal{L}$ be any line arrangement on $\mathbb{R}^2$. Consider the regions cut out by $\mathcal{L}$. For a fixed $u \in \mathbb{R}^2$, let a_k be the number of regions such that the projection dimension from u is equal to k. Prove that a_k is a constant if u is outside a finite union of lines. And the characteristic polynomial $\chi_{\mathcal{L}}(t)$ is $t^2 - a_1 t + a_0$.*
3. *Prove similar result for plane arrangements in $\mathbb{R}^3$, or general $\mathbb{R}^d$.*
4. *Use this result to reprove Zaslavsky's theorem.*

Project 3. *In this project, you will investigate the properties of torus arrangements.*

Let $T^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the set of complex numbers with unit length. Similarly, we define

$$T^d = \{(z_1, z_2, \cdots, z_d) \mid z_i \in \mathbb{C}, |z_i| = 1\}$$

A hypertorus H is defined by equation

$$z_1^{a_1} \cdots z_n^{a_d} = c$$

for integers a_i and complex number c with unit length. Here $(a_1, \cdots, a_d) \neq (0, 0, \cdots, 0)$.

1. *Show by example that H may not be connected.*
2. *Show by example that when two hypertoruses H_1 and H_2 are connected, their intersection may not be connected.*
3. *Find a condition for H to be connected. (Hint: if the greatest common divisor of $a_1, \cdots, a_d$ is 1, then the vector $\vec{n} = (a_1, a_2, \cdots, a_d)$ can be extended to an invertible integer matrix with $\vec{n}$ as the first column.)*
4. *Construct a bijection between disjoint union of several copies of T^{d-1} and H.*
5. *Consider a collection $\mathcal{H}$ of finitely many hypertoruses H_i in T^d, construct an intersection poset $P(\mathcal{H})$ consisting of connected components of intersections of H^i. Is this poset a lattice? Is it graded?*
6. *Is the number of connected components in $T^d - \cup_i H_i$ still related to the characteristic polynomial of $P(\mathcal{H})$ the same as Zaslavsky theorem?*
7. *Let $G = (V, E)$ be a graph with d vertices $1, \cdots, d$. For each edge connecting i and j, define a hypertorus $H_{ij} = \{z_i = z_j\}$. This forms a collection of hypertorus $\mathcal{H}$. Is the chromatic polynomial still equal to the characteristic polynomial of $P(\mathcal{H})$?*
8. *Find a relation between the number of regions for graphic torus arrangement and the chromatic polynomial.*