ALGEBRAIC COMBINATORICS II, HOMEWORK 2
DUE AUGUST 1 AT 5:30PM

Some ground rules:

- Feel free to use English, Chinese, or both, in your solutions.
- Write your argument as clear as possible, and make sure the writing in your submission is clear.
- Feel free to use results that are proved in class. If you'd like to use other results, you have to prove them before using them.
- You're encouraged to work together on the assignments. In your solutions, you should acknowledge the students with whom you worked, and should **write solutions on your own**.

Problems:

(1) Prove that to "give a map

$$G \times X \to X; \quad (g, x) \mapsto g \cdot x \qquad \text{satisfying:}$$

- $e_G \cdot x = x$ for any $x \in X$,
- $g \cdot (h \cdot x) = (gh) \cdot x$ for any $g, h \in G$ and $x \in X$;"

is equivalent to "give a group homomorphism $G \to S_X$".

(2) For $n \geq 3$, prove that for any non-identity element $\sigma \in S_n$, there exists $\tau \in S_n$ such that $\sigma\tau \neq \tau\sigma$. (Hint: Since $\sigma \neq e$, there exists $i, j \in \{1, \ldots, n\}$ such that $\sigma(i) \neq j$. Since $n \geq 3$, there exists an element $k \in \{1, \ldots, n\}$ different from i and j. Now, consider $\tau = (ik)$.)

(3) In this exercise, we prove more properties of symmetric groups and alternating groups.

(a) Prove that the transpositions $(12), (13), \ldots, (1n)$ together generate S_n. (Hint: $(ab) = (1a)(1b)(1a)$.)

(b) Prove that for $n \geq 3$, the 3-cycles generate A_n. In other words, any element in the alternating group can be written as $(a_1b_1c_1) \cdots (a_kb_kc_k)$. (Hint: $(1a)(1b) = (1ba)$.)

(c) Let $H \subseteq S_n$ be an index 2 subgroup. Prove that $H = A_n$. (Hint: Suppose $(123) \notin H$, show that then H, $(123)H$, $(132)H$ are three disjoint left cosets of H.)

(4) Prove the following relations between the whole symmetry groups $\mathrm{Aut}(-)$ and the rotational symmetry groups $\mathrm{Aut}^+(-)$.

(a) Let T be a tetrahedron. Prove that $\mathrm{Aut}(T) \not\cong \mathrm{Aut}^+(T) \times (\mathbb{Z}/2\mathbb{Z})$.

(b) Let C be a cube. Prove that $\mathrm{Aut}(C) \cong \mathrm{Aut}^+(C) \times (\mathbb{Z}/2\mathbb{Z})$.

(5) We are given a necklace with n beads, arranged in a cycle. We would like to put a color on each of the beads. Suppose there are m available colors that we can use.

(a) What is the number of distinct necklaces, counted up to rotations?

(b) What is the number of distinct necklaces, counted up to rotations and reflections?