ALGEBRAIC COMBINATORICS II, HOMEWORK 4
DUE AUGUST 15 AT 5:30PM

Some ground rules:

- Feel free to use English, Chinese, or both, in your solutions.
- Write your argument as clear as possible, and make sure the writing in your submission is clear.
- Feel free to use results that are proved in class. If you'd like to use other results, you have to prove them before using them.
- You're encouraged to work together on the assignments. In your solutions, you should acknowledge the students with whom you worked, and should **write solutions on your own**.

Problems:

(1) Recall that affine maps $z \mapsto Az + B$ and the inversion $z \mapsto \frac{1}{z}$ can both be considered as bijective continuous maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

(a) Prove that the affine maps take circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$.

(b) Prove that the inversion takes circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$.

Recall that circles in $\mathbb{C}$ and straight lines in $\mathbb{C}$ are both considered as circles in $\hat{\mathbb{C}}$.

(2) Consider the standard basis $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$ of $\mathbb{R}^3$, and denote $R_i(\theta)$ the counterclockwise rotation of angle θ with respect to the $\vec{e}_i$-axis. Show that any rotation $A \in \mathrm{SO}(3,\mathbb{R})$ can be written as a composition $R_1(\theta_1)R_2(\theta_2)R_3(\theta_3)$ for some $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$.

(3) Find a Möbius transformation that maps the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ onto the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, and sends 0 to i.

(4) Let ρ_1 and ρ_2 be two non-identity Möbius transformations with a common fixed point z_0 (they might have other fixed points which could be distinct). Prove that the Möbius transformation

$$\rho_1 \circ \rho_2 \circ \rho_1^{-1} \circ \rho_2^{-1}$$

is either parabolic or the identity.

(5) Let ρ_1 and ρ_2 be two non-identity Möbius transformations.

(a) Prove that if the fixed point sets of ρ_1 and ρ_2 are the same, then $\rho_1\rho_2 = \rho_2\rho_1$.

(b) Does the converse hold? In other words, prove or disprove the following statement: "if $\rho_1\rho_2 = \rho_2\rho_1$, then the fixed point sets of ρ_1 and ρ_2 are the same". (Note that ρ_1 and ρ_2 are assumed to be non-identity.)