Already in ancient times, we see evidence that people knew how to solve quadratic equations like $x^2 - x - 1 = 0$. More than 2500 years ago, mathematicians figured out that solutions to such equations were often irrational numbers, that is cannot be written as quotients a/b of integers. An important feature of the resolution of quadratic equations with rational coefficients is that the method goes through in exactly the same way whatever the quadratic equation it is applied to and that the solutions it returns appear to be similar: both solutions are rational numbers or neither of them is, both solutions are real numbers or neither of them is, in all cases the solutions seem to be symmetric in some sense and using number theory, we will show that there is essentially one way to express them.

What happens when we move to cubic equations like $x^3 - x - 1 = 0$ or $x^3 - 3x - 1 = 0$? Surprisingly, all the previous properties seem to fail. To begin with, devising a method to solving such equations proves to be a difficult task (don't worry, we will succeed in this task by uncovering a path that makes the solution almost obvious). With this method in hand, we will stumble on a perhaps nasty surprise: now solutions appear to go all over the place, with weird happenstance after weird happenstance occurring. For instance, solving the two cubic equations above leads to very different outcomes despite the apparent close similarity between them. Furthermore, honest real solutions sometimes appear disguised as square root of negative numbers.

Mathematicians struggled for 300 years to make sense of these phenomena, but it was well worth the effort: the solution of the mystery proved to be a link between the theory of equations and the notion of symmetry provided by mathematical structures hidden in the resolution of equations. The aim of this course is to reveal these hidden symmetries and structures using a blend of abstract algebra, arithmetic, clever algebraic manipulations of polynomial and even combinatorics and geometry. Mastering these symmetries allow for wonderful achievements of mathematics that can only be dreamed of without them. Among them:

1. Proving that certain constructions cannot be performed using a ruler and compass.
2. Computing exact algebraic expressions for numbers like $\cos\frac{2\pi}{5}$, $\cos\frac{2\pi}{7}$ and $\cos\frac{2\pi}{17}$.
3. Proving that one cannot write $\sqrt{11}$ using only rational numbers and $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and $\sqrt{7}$.
4. Explaining why it is impossible to do mathematics without complex numbers.
5. Finding all the numbers that can be expressed using the roots of $x^4 - 2$.

Discovering them will also offer a glimpse in questions at the current edge of mathematical research.

Prerequisites for this course include high-school algebra and the basic of calculus with polynomials. Some notions of geometry and linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ and/or prior knowledge of complex numbers is helpful, but not required.