Homework 1

This homework set must be handed in by Monday, July the 29th after the tutorial session. You don't need to try to do everything. We will value much more an incomplete homework that has been done perfectly than a homework that claims to be complete but which does not rise up to mathematical standards.

The point of this homework is to assess your progress as aspiring mathematicians. Consequently, you are free to work collectively and you are welcome to ask us for any help you might need to improve your answers. Every answer should contain all necessary arguments. In particular, a question that asks you to prove something can very rarely be answered solely by a computation. Any answer in the version you hand in which is logically false or logically incomplete will be marked zero. For instance, if you used the fact that a number a is non-zero (to divide by a, say) but forgot to state it explicitly as a hypothesis, your answer will be marked zero even if your proof was perfect for non-zero numbers; if you were asked to prove that P is a polynomial with real, positive coefficients and you correctly proved that P has positive coefficients but you didn't explicitly check before that that P has real coefficients and thus that the question makes sense, your answer will be marked zero (even if it is obvious that the coefficients of P are real); if you implicitly used a non-trivial property (the Fundamental Theorem of Algebra, for instance) and failed to mention it explicitly, your answer will be marked zero (even if you have correctly checked that all the hypotheses required to use this property are indeed verified); if your argument is a proof by contradiction but you didn't explicitly state what is the false hypothesis and/or explicitly point out where is the contradiction, your answer will be marked zero. You are allowed to provide incomplete answers provided you make it clear that you don't claim a full answer. For instance, if you are asked to prove that A is equivalent to B and you only manage to prove that A implies B, you will get points. If you correctly proved A implies B but claim you have proved A is equivalent to B, your answer will be marked zero. if you write the word *obviously*, your answer will marked -1, because anything that is really obvious has an easy proof that you should have provided instead of wasting your time and ours writing *obviously*. To avoid a negative grade, you are therefore strongly encouraged to make sure that your answers are up to the standards of mathematical rigor before handing in your homework.

Here are tips for good writing. Do state hypotheses before stating conclusions. Do make clear what are the objects you are considering and the sets they belong to (for instance, do write sentences like *Let* $D \in \mathbb{Q}[x]$ *be* $P \wedge Q$ rather than $D = P \wedge Q$). Do separate cases (for instance, do write sentences like *First assume that the degree d of P is less than or equal to 2. Then... Now we assume that d is strictly greater than 2. Then...*). Say out loud the sentence that you want to write. If you find out that you need to backtrack or that you are mentioning an object that you haven't introduced yet, start by writing preliminary sentences introducing what you realized you needed (for instance, if you realize that you need $P \wedge Q$, do start by writing *Let* D *be* $P \wedge Q$ or even better as we saw *Let* $D \in \mathbb{Q}[x]$ *be* $P \wedge Q$). Think of your proofs as a dialogue with a skeptical peer or a skeptical version of yourself: whenever you write an argument, imagine that this person is asking you *why?* or *how do you know that?*. If you realize the answer of this question involves a certain justification, put this justification before the sentence (for instance, if you were about to write α *is irrational*, ask yourself why, and if the answer that comes to your mind is *Because* α *is a complex number which is not real*, ask yourself why, and if the answer that comes to your mind is *Because* $\bar{\alpha} \neq \alpha$, write *We have seen that* $\bar{\alpha} \neq \alpha$ *so* α *is a complex number which is not real so* α *is irrational*). Do find a skeptical peer to actually engage in such a dialogue about your proof and be that peer for someone else. Do write a preliminary version of the proof then start over to see if you can make it better and if you have indeed successfully established everything that was required.

Exercise 1. *Consider*

$$A = x^6 - x^5 + x^4 - 5x^3 + 11x^2 - 11x + 6, B = x^5 - 7x^4 + 25x^3 - 21x^2 + 15x + 3$$

1. *Find* $A \wedge B$.
2. *Give the decomposition of* B *into prime polynomials in* $\mathbb{Q}[x]$.
3. *In question 1, it was not made precise whether* A *and* B *are polynomials in* $\mathbb{Q}[x], \mathbb{R}[x]$ *or* $\mathbb{C}[x]$. *Does the answer of the question depend on this choice? Explain why or why not.*

Exercise 2.

1. *Prove that there are infinitely many primes in* $\mathbb{Z}$.
2. *Copy your previous proof to prove that there are infinitely many prime polynomials in* $K[x]$.

Exercise 3. *Let* $P \in \mathbb{Q}[x]$ *be a polynomial. Let* $\beta \in \mathbb{C}$ *be* $P(i\sqrt{5})$.

1. *Show that* β *can be written in a unique way* $a + bi\sqrt{5}$ *with* $(a, b) \in \mathbb{Q}^2$ *and give a description of* a *and* b *in terms of the polynomials* P *and* $x^2 + 5$.
2. *State a generalization of what you have proven.*

Problem 1. *The aim of this problem is to describe completely the set of prime polynomials of* $\mathbb{R}[x]$. *Let* $P \in \mathbb{R}[x]$ *be such a prime polynomial.*

1. *Describe* P *if* P *is of degree 1.*

Suppose now that $\deg P > 1$

2. *Show that* P *has a complex root* $z_1 \in \mathbb{C}$ *which is not in* $\mathbb{R}$.
3. *Prove that* P *has another complex root* z_2 *satisfying the same property (*z_2 *is complex but not real).*
4. *Prove that* $(x - z_1)(x - z_2) \in \mathbb{C}[x]$ *is a polynomial with coefficients in* $\mathbb{R}$ *which divides* P *in* $\mathbb{R}[x]$.
5. *Deduce that* $\deg P = 2$.

We write $P = x^2 + bx + c$.

6. *Show that* $b^2 - 4c < 0$.
7. *Show conversely that if* $P = x^2 + bx + c \in \mathbb{R}[x]$ *with* $b^2 - 4c < 0$, *then* P *is a prime polynomial.*
8. *Conclude by giving a complete description of the set of prime polynomials of* $\mathbb{R}[x]$.