Homework 2

Exercise 1. *Find all the solutions to the following equations.*

1. $x^3 - 6x - 6 = 0$ *and* $x^3 - 3x - 4 = 0$.
2. $x^3 + 6x - 20 = 0$ *and* $x^3 - 3x + 2 = 0$. *Show that* $\sqrt[3]{10 + 6\sqrt{3}} = 1 + \sqrt{3}$. *What is going on?*
3. $x^3 - 3x + 1 = 0$. *Find another solution using* $\cos$.
4. $x^3 + 2x^2 - 5x + 1 = 0$. *Show that all solutions are real numbers.*

Exercise 2. *Let p be a prime.*

1. *Show using the p-adic valuation that*
$$p \mid \binom{p}{r}$$
if $1 \leq r \leq p - 1$.
2. *Show that*
$$\frac{(x+1)^p - 1}{x}$$
is irreducible in $\mathbb{Q}[x]$.
3. *Deduce that Φ_p is the minimal polynomial of ζ_p in* $\mathbb{Q}[x]$.

Problem 1. *Recall that* $\varphi(n) = |\mathbb{U}'_n| = \deg \Phi_n$.

1. *Let p be a prime number. Show that* $\varphi(p) = p - 1$.

Let $\zeta \in \mathbb{C}$ be a root of unity of exact order n and let $\xi \in \mathbb{C}$ be a root of unity of exact order m with $n \wedge m = 1$.

2. *Show that $\zeta\xi$ is a root of unity of exact order nm.*
3. (a) *Show that $x \mapsto x^m$ is a bijective map from $\mathbb{U}'_n$ to itself.*
   (b) *Deduce from that that any $\psi \in \mathbb{U}'_{nm}$ can be written $\psi = \zeta\xi$ with* $(\zeta, \xi) \in \mathbb{U}'_n \times \mathbb{U}'_m$.
4. *Show by giving counterexamples that all the statements above can definitely be false without the hypothesis* $n \wedge m = 1$.
5. *Conclude that $\varphi(nm) = \varphi(n)\varphi(m)$ if* $n \wedge m = 1$.
6. *Compute* $\varphi(2024)$.