Homework 3

Conventions We write 1_n for a generator of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. If $x \in \mathbb{N}$, we write x_n for the sum $1_n + \cdots + 1_n$ repeated x times. If x is a negative relative number, we write $-x_n$ for the opposite of x_n in $\mathbb{Z}/n\mathbb{Z}$.

Problem 1 (Cyclic groups). *1. Find all the generators of the cyclic group $\mathbb{Z}/15\mathbb{Z}$.*

2. *What is the image of $\mathbb{U}'_n$ in an isomorphism between $\mathbb{U}_n$ and $\mathbb{Z}/n\mathbb{Z}$?*
3. *Draw the lattice of all subgroups of $\mathbb{Z}/7\mathbb{Z}$, $\mathbb{Z}/16\mathbb{Z}$ and $\mathbb{Z}/30\mathbb{Z}$. List explicitly the elements of each subgroups (it is OK to use the last question of this problem to justify your answer).*
4. *Let G be a finite cyclic group. Let $a \in G$ be a generator (by definition, this means that $\langle a \rangle = G$).*
   (a) *Show that $H_d \overset{\text{def}}{=} \langle a^{n/d} \rangle$ is a subgroup of G. What is $|H_d|$?*
   (b) *Let $H'_d \subset G$ be a subgroup of cardinality d.*
      i. *Show that $\{x \in \mathbb{Z} | a^x \in H'_d\}$ is a non-zero subgroup of $\mathbb{Z}$. Let y be its smallest strictly positive element.*
      ii. *Show that $\langle a^y \rangle = H'_d$.*
      iii. *En déduire que $H_d = H'_d$.*
5. *Conclude that all subgroups of finite cyclic groups are finite cyclic groups and that a finite cyclic group of cardinality n has exactly one subgroup of cardinality d a divisor of n.*

Problem 2 (Product of groups). *If G and H are groups with group operations $\cdot$ and $*$ respectively, we consider the set*

$$G \times H \overset{\text{def}}{=} \{(g, h) | g \in G, h \in H\}$$

with the operation $\circ$ defined by

$$(g, h) \circ (g', h') \overset{\text{def}}{=} (g \cdot g', h * h').$$

1. *The aim of this question is to show that $G \times H$ with $\circ$ is a group.*
   (a) *Briefly show that $\circ$ is associative (any answer which is more than 4 lines is wrong).*
   (b) *What the identity element of $G \times H$? What is the inverse of (g, h) for $\circ$?*
2. *What is the cardinality of $G \times H$ (assuming that G and H are finite groups)?*
3. *Let $(n, m) \in \mathbb{N}^2$ be two non-zero numbers. We study the group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.*
   (a) *Give an explicit of elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Are these groups cyclic? If so, find a generator. Is $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ isomorphic to $\mathbb{Z}/9\mathbb{Z}$?*
   (b) *Assume that $n \wedge m = 1$. Show that $x_{nm} \longmapsto (x_n, x_m)$ is an isomorphism of groups from $\mathbb{Z}/nm\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.*
4. *Give a group-theoretic proof of the fact that $(n!m!) \mid (n+m)!$ (hint: think of the permutation action on $\{1, \cdots, n+m\}$ and of the stabilizer of $\{1, \cdots, n\}$).*
5. *Write the lattice of subgroups of $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.*

Problem 3 (Dihedral groups)**.** *Let $n \geq 2$ be an integer. Let X be the unit circle, that is to say the set*

$$X \overset{\text{def}}{=} \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\} = \{z \in \mathbb{C} | z\bar{z} = 1\}.$$

Let Γ be the group of isometries of $\mathbb{R}^2$ (we admit that this is a group). Let D_{2n} be the stabilizer in Γ of $\mathbb{U}_n$.

1. (a) *Show that D_{2n} is a group*
   (b) *Show that*
   $$\sigma_n : z \longmapsto e^{\frac{2\pi i}{n}} z, \ \tau : z \longmapsto \bar{z}.$$
   belong to D_{2n}.
   (c) *Is D_{2n} a commutative group? (Be careful and precise.)*
   (d) *What are the cardinality of $\langle \sigma_n \rangle$ and $\langle \tau \rangle$?*
   (e) *Show by induction that if $\gamma \in D_{2n}$ sends 1 to 1 and $e^{\frac{2\pi i}{n}}$ to $e^{\frac{2\pi i}{n}}$, then it is the identity.*
   (f) *Show that any $\gamma \in D_{2n}$ can be written $\gamma = \sigma_n^r$ or $\gamma = \tau\sigma_n^r$ for some $r \in \mathbb{Z}/n\mathbb{Z}$.*
   (g) *Explain how we can recognize whether $g = \sigma^r$ or $g = \tau\sigma^r$ geometrically in terms of action of g on $\mathbb{U}_n$.*
2. *Deduce $|D_{2n}|$ from all sub-questions of the previous question.*
3. *Let g be an element of D_{2n}.*
   (a) *Suppose $g = \tau\sigma^r$. What is $|\langle g \rangle|$?*
   (b) *Suppose now that $g = \sigma^r$. What is $|\langle g \rangle|$?*
4. *Show that D_6 is isomorphic to $\mathfrak{S}_3$.*
5. *Consider $n \geq 3$. Show that D_{2n} is isomorphic to a subgroup of $\mathfrak{S}_n$ by constructing an explicit isomorphism.*
6. *Write the lattice of subgroups of D_6 (have you seen this lattice in your life before?).*