Homework 3

Exercise 1. 1. *Show that any quadratic extension $K(\sqrt{d})$ is a Galois extension of K.*

2. *Find* $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ *assuming that Φ_n is the minimal polynomial of ζ_n.*
3. *Find an example of $\mathbb{Q} \subset K \subset L$ such that $\mathbb{Q} \subset K$ is a Galois extension and $K \subset L$ is a Galois extension but $\mathbb{Q} \subset L$ is not a Galois extension (hint: use the first question).*
4. *Show that it is impossible to write ζ_5 as the sum $a+b\sqrt{n}+c\sqrt{m}+d\sqrt{nm}$ with $(n,m) \in \mathbb{Z}^2$.*
5. *Show that $\mathbb{Q}(\jmath, \sqrt[3]{2})$ is a Galois extension of $\mathbb{Q}$ and find its Galois group.*
6. *Find an example of $\mathbb{Q} \subset K \subset L$ such that $\mathbb{Q} \subset L$ is a Galois extension and $K \subset L$ is a Galois extension but $\mathbb{Q} \subset K$ is not a Galois extension.*
7. *Find the lattice of sub-extensions of $\mathbb{Q}(\zeta_{13})$.*

Exercise 2. *Let $p_1, \cdots, p_r$ be distinct primes. Let $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \cdots, \sqrt{p_r})$.*

1. *Give an obvious upper bound for $[K : \mathbb{Q}]$.*
2. *Show that $K/\mathbb{Q}$ is a Galois extension.*
3. *Show that* $\mathrm{Gal}(K/\mathbb{Q})$ *is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ for some $m \in \mathbb{N}$.*
4. *Find an obvious lower bound for the number of quadratic sub-extensions of K.*
5. *Conclude (notice that no computations was required).*

Exercise 3. *Consider $K \subset \mathbb{C}$. Let $P = a_n x^n + \cdots + a_0 \in K[x]$ be a polynomial whose complex roots are $\{\alpha_1, \cdots, \alpha_n\}$. Suppose $L = K(\alpha_1, \cdots, \alpha_n)$.*

1. *Show that L/K is a Galois extension.*
2. *Show* $\mathrm{Gal}(L/K)$ *acts faithfully on $\{\alpha_1, \cdots, \alpha_n\}$.*
3. *Deduce that* $\mathrm{Gal}(L/K)$ *can be seen as a subgroup of $\mathfrak{S}_n$.*
4. *Find a condition on P which is equivalent to the action of* $\mathrm{Gal}(L/K)$ *being transitive.*

We define

$$\mathrm{disc}(P) = a_n^{2n-2} \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

5. *Show that*

$$\mathrm{disc}(P) = (-1)^{n(n-1)/2} a_n^{n-2} \prod_{i=1}^{n} P'(\alpha_i)$$

6. *Compute* $\mathrm{disc}\left(ax^2 + bx + c\right)$ *and* $\mathrm{disc}(x^3 + px + q)$.
7. *Suppose P is monic. Show that* $\mathrm{Gal}(L/K)$ *viewed as a subgroup of $\mathfrak{S}_n$ is a subgroup of A_n if and only if* $\mathrm{disc}(P)$ *is a square in K.*
8. *Find a polynomial P as above such that* $\mathrm{Gal}(L/K)$ *is $\mathbb{Z}/3\mathbb{Z}$. Solve the equation $P(x) = 0$. What do you observe?*