ALGEBRA AND TOPOLOGY
HOMEWORK FIVE
DUE: 8/7

There are two different types of labels: alphabets and numbers. You only need to write up your solutions to those exercises labelled by alphabets. The rest is for fun.

Exercise A. Let $(G, *)$ be a group. Show that

(1) the identity element $e \in G$ is unique;
(2) the inverse element of an element $a \in G$ is unique. (Therefore, the inverse of a will be denoted by a^{-1}.)
(3) Let $H \subseteq G$ be a subset. We say that H is a subgroup of G is H is also a group under the binary operation $*$ in G. Show that H is a subgroup of G if and only if $a * b^{-1} \in H$ for all $a, b \in H$.
(4) Given a subgroup H, we can define a relation on G by "$a \sim b$ if and only if $a * b^{-1} \in H$." Show that this is an equivalence relation on G. Recall that an equivalence relation $\sim$ on a set S is a relation with the following properties.
  (EQ1) (reflexivity) $a \sim a$;
  (EQ2) (symmetry) $a \sim b$ if and only if $b \sim a$;
  (EQ3) (transivity) $a \sim b$ and $b \sim c$ imply $a \sim c$.

Hint: To show that the identity element is unique, it suffices to show that if e and e' are identity elements, then $e = e'$. Similarly, to prove the uniqueness of inverse elements, you have to show that for every $a \in G$ and if b and b' are inverse elements of a, then $b = b'$.

Exercise B. Let $(G, *)$ be a group. A subgroup H is called *a normal subgroup*, if $a^{-1} * H * a \subset H$ for every $a \in G$. Show that the set of equivalence classes (or the set of cosets)

$$G/H := \{aH \mid a \in G\}$$

where $aH := \{a * h \in G \mid a \in G \text{ and } h \in H\}$ is a group under the operation

$$(aH) \star (bH) := (a * b)H.$$

Show that if G is abelian, then every subgroup is normal.
Hint: You have to check that the operation $\star$ is well-defined.

Exercise 1. Consider $S = \mathbf{R}^2 \setminus \{(0,0)\}$. Define a binary operation via

$$(a, b) \bullet (c, d) := (ac - bd, ad + bc).$$

Show that $(S, \bullet)$ is a group.