ALGEBRA AND TOPOLOGY
HOMEWORK SEVEN
DUE: 8/12

There are two different types of labels: alphabets and numbers. You will be asked to write up your solutions to those exercises labelled by alphabets $A, B, \ldots$ only. The rest is for fun.

Exercise A. Let $\mathcal{K}$ be a directed complex and consider the integral homology theory on $\mathcal{K}$.

(a) Give the definition of incidence coefficients and incidence matrices in integral homology theory.

(b) Show that $\partial_1 \circ \partial_2 = 0$ on any complex $\mathcal{K}$.

Exercise B. Show that if a surface S is connected, then $b_0(S; \mathbb{F}_2) := \dim \mathrm{H}_0(S; \mathbb{F}_2) = 1$. Moreover, if S is sequentially compact, then we have $b_2(S; \mathbb{F}_2) = \dim \mathrm{H}_2(S; \mathbb{F}_2) = 1$. Conclude that for any sequentially compact and connected surface S, the Euler characteristric is given by

$$\chi(S) = 2 - \dim \mathrm{H}_1(S; \mathbb{F}_2).$$

Exercise 1. Calculate the integral homology groups for Klein bottle K. Moreover, what are the integral Betti numbers? Are they the same as the Betti numbers $b_k(K; \mathbb{F}_2)$? How about the Euler characteristics?

Exercise 2. Consider the group

$$G := \{A \in \mathrm{Mat}_{3\times 3}(\mathbf{R}) \mid \det(A) = 1 \text{ and } A^\intercal A = AA^\intercal = \mathrm{I}_3\}$$

and a subset

$$H := \left\{ A \in G \;\middle|\; H \text{ is of the form } \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Show that H is a subgroup. Can you describe the quotient (cotset space) $S := G/H$ geometrically? Certainly G is equipped with a topology inherited from the metric topology on $\mathbf{R}^9$ (the induced topology). Calculate the integral homology groups for S.