ALGEBRA AND TOPOLOGY
HOMEWORK SIX
DUE: 8/9

There are two different types of labels: alphabets and numbers. You only need to write up your solutions to those exercises labelled by alphabets. The rest is for fun.

Exercise A. Show that the map $\mathrm{H}_1(\mathcal{K};\mathbb{F}_2) \to \mathrm{H}_1(\mathcal{K}^+;\mathbb{F}_2)$ is injective.

Exercise B. Show that the homology groups remain the same when we perform Step II. Therefore we conclude the proof of the invariance theorem.

Exercise 1. Let $T_i \colon V_i \to V_{i-1}$ be a finite sequence of linear maps between vector spaces over a field F indexed by $\mathbf{Z}$ such that $T_i \circ T_{i+1} = 0$ for all i. Here, by a finite sequence, we mean there exists an $m \geq 0$ such that $V_i = 0$ for every $|i| \geq m$.

Let $S_j \colon W_j \to W_{j-1}$ be another finite sequence of linear maps between vector spaces indexed by $\mathbf{Z}$ such that $S_j \circ S_{j+1} = 0$ for all j.

Suppose that there are linear maps $\phi_k \colon V_k \to W_k$ such that all the squares in the following diagram commute.

$$\begin{array}{ccccc} V_{i+1} & \xrightarrow{T_{i+1}} & V_i & \xrightarrow{T_i} & V_{i-1} \\ \downarrow{\scriptstyle \phi_{i+1}} & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i-1}} \\ W_{i+1} & \xrightarrow{S_{i+1}} & W_i & \xrightarrow{S_i} & W_{i-1} \end{array}$$

In other words, $\phi_{i-1} \circ T_i = S_i \circ \phi_i$ and $\phi_i \circ T_{i+1} = S_{i+1} \circ \phi_{i+1}$. Show that the linear map ϕ_i induces a linear map between the quotient spaces

$$\mathrm{Ker}(T_i)/\mathrm{Im}(T_{i+1}) \to \mathrm{Ker}(S_i)/\mathrm{Im}(S_{i+1}),\ v + \mathrm{Im}(T_{i+1}) \mapsto \phi_i(v) + \mathrm{Im}(S_{i+1}).$$

Exercise 2. The aim of this exercise is to show that every finitely generated abelian group G is isomorphic to

$$\mathbf{Z}^n \oplus \mathbf{Z}/d_1\mathbf{Z} \oplus \cdots \mathbf{Z}/d_m\mathbf{Z}$$

with $d_i \mid d_{i+1}$ for $i = 1, \ldots, m-1$.

(a) Show that every subgroup of a finitely generated abelian group is finitely generated. (**Hint**: induction on the number of generators.)

(b) From (a), show that every finitely generated abelian group G is isomorphic to $\mathbf{Z}^r/\mathrm{im}(A)$, where

$$A \colon \mathbf{Z}^q \to \mathbf{Z}^r$$

is an integral matrix.

(c) Show that there exist *invertible* matrices $P \in \mathrm{Mat}_{r\times r}(\mathbf{Z})$ and $Q \in \mathrm{Mat}_{q\times q}(\mathbf{Z})$ such that

$$PAQ = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & d_m & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathrm{Mat}_{r\times q}(\mathbf{Z})$$

(d) Conclude the result from (c).