ALGEBRA AND TOPOLOGY
HOMEWORK TWO
DUE: 7/29

There are two different types of labels: alphabets and numbers. You only need to write up your solutions to those exercises labelled by alphabets. The rest is for fun.

Exercise A. Show that sequential compactness is a toplogical property by estabilishing the following stronger statement. Let $f\colon X \to Y$ be a continuous transformation. If X is sequentially compact, then $f(X)$ is sequentially compact.

P.S. I made a mistake during my class. Please fix it by yourself.

Exercise B. Show that (S^2, τ) defined in the lecture is a topological space.

Exercise 1. Suppose that f is a transformation having the property that whenever $p \leftarrow \{a_m\}_{m=1}^{\infty}$, we have $f(p) \leftarrow \{f(a_m)\}_{m=1}^{\infty}$. Show that f is continuous.

Exercise 2. Recall that the Bolzano–Weierstrass theorem states that for a subset $S \subset \mathbf{R}^2$, the set S is sequentially compact if and only if S is closed and bounded. Here is an outline of the proof of Bolzano–Weierstrass theorem. In this exercise, you should complete each step and hence conclude a proof.

Let S be a closed and bounded subset in $\mathbf{R}^2$. Take a sequence $\mathcal{A} := \{a_m\}_{m=1}^{\infty}$ in S. We will show that $\mathcal{A}$ has a near point in S.

- Since S is bounded, S is contained in a rectangle. Generalize the "bisection" technique we used for the proof of connectedness in class to show that $\mathcal{A}$ contains a near point in $\mathbf{R}^2$.
- Use the closedness to conclude that the point you found in preceding bullet is contained in S.

Where do you use the fact that $\mathbf{R}$ is complete?

Exercise 3. Let (S, τ) be a topological space.

- Write down the definition for a subset $D \subset S$ being sequentially compact.
- Write down the definition for a subset $D \subset S$ being connectedness.
- Let (T, σ) be another topological space. Let $f\colon S \to T$ be a map between sets S and T. Write down the definition of f being continuous.