ALGEBRA AND TOPOLOGY RESEARCH PROJECTS

This is a preliminary version, subject to change.

Project A. Let $S^n := \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = 1\}$ be the unit sphere equipped with the metric topology. For $p = (y_0, \dots, y_n) \in S^n$, we denote by $-p$ the point $(-y_0, \dots, -y_n)$.

(a) Let $f\colon S^1 \to \mathbf{R}^m$ be a continuous transformation. Can you always find a point $p \in S^1$ such that $f(p) = f(-p)$? Justify your answer.

(b) Let $f\colon S^2 \to \mathbf{R}^m$ be a continuous transformation. Can you always find a point $p \in S^2$ such that $f(p) = f(-p)$? Justify your answer.

(c) Solve the same problem for $f\colon S^n \to \mathbf{R}^m$.

Project B. We introduced homology groups for complexes and surfaces. This project is about the homology groups.

(a) Classify all possible finitely generated abelian groups G that can be realized as the 0^{th} homology of a connected complex.

(b) Classify all possible finitely generated abelian groups G that can be realized as the 1^{st} homology of a connected complex.

(c) Classify all possible finitely generated abelian groups G that can be realized as the 2^{nd} homology of a connected complex.

(d) Re-do the problems (a), (b), and (c) for connected *surfaces*.

Project C. This project is related to the map coloring problem.

A *map* is simply a complex in which the polygons are called *countries*. Using this dictionary, we can state the famous "map coloring problem."

Question: What is the smallest number of colors needed to color all countries on the map such that any two countries sharing the same border have different colors?

We can easily generalize the problem to arbitrary surfaces as follows. Let S be a surface and $\mathcal{K}$ be a complex on S. A polygon in $\mathcal{K}$ will also be called a country. Denote by $n_S(\mathcal{K})$ the smallest number of colors needed to color all countries in $\mathcal{K}$ such that any two countries sharing the same border have different colors and

$$n_S := \max_{\mathcal{K}} \{n_S(\mathcal{K})\}$$

where the maximum is taken over all complexes on S.

(a) Show that if $S = S^2$ is the sphere, then $n_S = n_{\mathbf{R}^2}$.
(b) Show that $n_{\mathbf{R}P^2} \leq 6$.
(c) Let χ be the Euler characteristic of S. Can you estimate n_S using χ?
(d) Find n_K, where K is the Klein bottle.