Multidimensional problems: Minkowski Convex Body Theorem (Lecture 10, 1 August 2024).

1. **Blichfeldt's Lemma.** *Consider a set $A \subset \mathbb{R}^n$ of volume $\operatorname{vol} A > 1$. There exist two different points $x, y \in A$ such that $x - y \in \mathbb{Z}^n$.*

2. **Minkowski convex body theorem.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex 0-symmetric set of the volume $\operatorname{vol}\Omega > 2^n$. Then there exists a non-zero integer point which belongs to Ω.*

3. Mordell's proof of Minkowski Theorem: application of pigeon-hole principle to the set of rational points from

$$\frac{1}{q}\mathbb{Z}^n \cap \Omega.$$

Exercises.

1. Dirichlet again.

a. Simultaneous approximation. (If somebody is afraid he can assume that $n = 2$.)
a1. Let $\alpha_1, ..., \alpha_n$ be real numbers and $Q \in \mathbb{Z}_+$. Prove that

$$\min_{q \in \mathbb{Z}_+ :\, q \le Q^n} \max_{1 \le j \le n} ||q\alpha_j|| \le \frac{1}{Q}.$$

a2. Let $\alpha_1, ..., \alpha_n$ be real numbers and not all of them are rational. Prove that there exists infinitely many $q \in \mathbb{Z}_+$ such that

$$\max_{1 \le j \le n} ||q\alpha_j|| \le \frac{1}{q^{1/n}}.$$

b. Linear form. (If somebody is afraid he can suppose that $m = 2$.)
b1. Let $\alpha_1, ..., \alpha_m$ be real numbers and $Q \in \mathbb{Z}_+$. Prove that

$$\min_{q_1, ..., q_n \in \mathbb{Z} : 1 \le \max_j |q_j| \le Q} ||q_1\alpha_1 + ... + q_m\alpha_m|| \le \frac{1}{Q^n}.$$

b2. Let $1, \alpha_1, ..., \alpha_m$ be linearly independent over $\mathbb{Z}$, that is

$$q_0 + q_1\alpha_1 + ... + q_m\alpha_m \ne 0 \quad \forall (q_0, q_1, ..., q_m) \in \mathbb{Z}^{m+1} \setminus \{(0, 0, ..., 0)\}.$$

Prove that there exist infinitely many vectors $(q_1, ..., q_m) \in \mathbb{Z}^{m+1} \setminus \{(0, 0, ..., 0)\}$ such that

$$||q_1\alpha_1 + ... + q_m\alpha_m|| \le \frac{1}{(\max_{1 \le j \le m} |q_j|)^m}.$$

2. Irrationality measure functions.
a. Simultaneous approximation. Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$. Prove that

$$\psi_{\boldsymbol{\alpha}}(t) = \min_{q \in \mathbb{Z}_+ : q \le t} \max_{j=1,...,n} ||q\alpha_j|| \le t^{-\frac{1}{n}} \quad \forall t \ge 1.$$

b. Linear form. Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m$. Prove that

$$\psi^*_{\boldsymbol{\alpha}}(t) = \min_{(q_1, ..., q_m) \in \mathbb{Z}^m \setminus \{(0, ..., 0)\} : \max_j |q_j| \le t} ||q_1\alpha_1 + ... + q_m\alpha_m|| \le t^{-m} \quad \forall\, t \ge 1.$$

c. Systems of linear forms. Let

$$\Theta = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,m} \\ \cdots & \cdots & \cdots \\ \theta_{n,1} & \cdots & \theta_{n,m} \end{pmatrix}$$

Prove that

$$\psi_\Theta(t) = \min_{(q_1,\dots,q_m)\in\mathbb{Z}^m\setminus\{(0,\dots,0)\}:\max_j |q_j|\le t} \ \max_{1\le j\le n} ||q_1\theta_{j,1} + ... + q_m\theta_{j,m}|| \le t^{-\frac{m}{n}}, \ \forall\, t \ge 1.$$

3. Theorem on linear forms. Consider linear forms

$$L_j(x) = L_j(x_1, ..., x_n) = \sum_{i=1}^{n} \alpha_{i,j}x_i, \quad 1 \le j \le n$$

with determinant $\Delta = \det(\alpha_{i,j})_{1\le i,j\le n}$. Suppose that positive $\varepsilon_j, 1 \le j \le n$ satisfy

$$\varepsilon_1 \cdots \varepsilon_n \ge |\Delta|.$$

Then the system of inequalities

$$|L_1(x)| \le \varepsilon_1, \qquad |L_j(x)| < \varepsilon_j, \ 2 \le j \le n$$

has a non-zero integer solution $x = (x_1, ..., x_n) \in \mathbb{Z}^n$.

4. Consider linear forms

$$L_j(x) = L_j(x_1, ..., x_m) = \sum_{i=1}^{m} \alpha_{i,j}x_i, \quad 1 \le j \le n$$

Prove that for any $X \ge 1$ there exists $x = (x_1, .., .x_m) \in \mathbb{Z}^m$ such that

$$\max_{1\le j\le n} ||L_j(x)|| \le X^{-\frac{m}{n}}, \quad 1 \le \max_{1\le i\le m} |x_i| \le X.$$

5. About Diophantine constant. Let $\alpha_1, ..., \alpha_n$ be real numbers

a. Prove that for any $M, t > 0$ the set

$$\Omega(M,T) = \{(x, y_1, ..., y_n) \in \mathbb{R}^{n+1} : \ t^{-n}|x| + nt \max_{1\le i\le n} |y_i - \alpha_i x| \le M\}$$

is convex, compact, 0-symmetric and has volume

$$\frac{(2M)^{n+1}}{(n+1)n^n}.$$

b. Prove that if $(x, y_1, ..., y_n) \in \Omega(M,T)$ and

$$t^{-n}|x| \ne t \max_{1\le i\le n} |y_i - \alpha_i x|,$$

then

$$|x| \left(\max_{1\le i\le n} |y_i - \alpha_i x| \right)^n < \left(\frac{M}{n+1} \right)^{n+1}.$$

c. Prove that there exist infinitely many $q \in \mathbb{Z}_+$ with

$$q^{\frac{1}{n}} \max_{1 \le i \le n} ||\alpha_i q|| \le \frac{n}{n+1}.$$

6. There exist infinitely many $q \in \mathbb{Z}_+$ such that

$$q \prod_{i=1}^{n} ||\alpha_i q|| < \frac{n!}{(n+1)^n}.$$