

## Farey tree (Lecture No. 3, 23.07.2024).

1. Hurwitz Theorem.

a. Mediant fraction. Consider two fractions  $\frac{a}{b}, \frac{c}{d}$ ,  $(a, b) = (c, d) = 1$ .

1) Show that the *mediant fraction*  $\frac{a+c}{b+d}$  is located between  $\frac{a}{b}$  and  $\frac{c}{d}$ ;

2) suppose that  $|\frac{a}{b} - \frac{c}{d}| = \frac{1}{bd}$ , then  $|\frac{a}{b} - \frac{a+c}{b+d}| = \frac{1}{b(b+d)}$  and  $|\frac{a+c}{b+d} - \frac{c}{d}| = \frac{1}{(b+d)d}$ .

b. Hurwitz.

1) **Theorem 1.** Suppose that  $\alpha \in [\frac{a}{b}, \frac{c}{d}]$  and  $|\frac{a}{b} - \frac{c}{d}| = \frac{1}{bd}$ . Then either  $|\alpha - \frac{a}{b}| \leq \frac{1}{\sqrt{5}b^2}$ , or  $|\alpha - \frac{c}{d}| \leq \frac{1}{\sqrt{5}d^2}$ , or  $|\alpha - \frac{a+c}{b+d}| \leq \frac{1}{\sqrt{5}(b+d)^2}$ .

2) **Theorem 2.** For any irrational  $\alpha \in \mathbb{R}$  there exist infinitely many  $\frac{a}{q}$  with  $|\alpha - \frac{a}{q}| < \frac{1}{\sqrt{5}q^2}$ .

3) Why the constant  $1/\sqrt{5}$  in Theorem 2 is optimal?

2. Basis of  $\mathbb{Z}^2$ . Consider two vectors  $e_1 = (a, b), e_2 = (c, d) \in \mathbb{Z}^2$ , with  $ad - bc = \pm 1$ . Then

a. each of the vectors is primitive, that is  $(a, b) = (c, d) = 1$ .

b. every integer point  $z \in \mathbb{Z}^2$  can be written in a form  $z = \lambda e_1 + \mu e_2$  with integer  $\lambda, \mu$ .

3. Stern-Brocot sequences. Define the sets  $F_n, n = 0, 1, 2, \dots$  by the following inductive procedure.

For  $n = 0$  put

$$F_0 = \{0, 1\} = \left\{ \frac{0}{1}, \frac{1}{1} \right\}.$$

If  $F_n$  is defined write it as a collection of increasing numbers:

$$0 = \xi_{0,n} < \xi_{1,n} < \dots < \xi_{N(n),n} = 1, N(n), \quad \xi_{j,n} = \frac{p_{j,n}}{q_{j,n}}, \quad (p_{j,n}, q_{j,n}) = 1.$$

Then we define  $F_{n+1}$  as

$$F_{n+1} = F_n \cup Q_{n+1}$$

where

$$Q_{n+1} = \left\{ \frac{p_{j,n} + p_{j+1,n}}{q_{j,n} + q_{j+1,n}}, j = 0, \dots, N(n) - 1 \right\}.$$

a. What is  $N(n)$ ?

b. Prove that for every rational  $\xi \in [0, 1]$  there exists  $n$  such that  $\xi \in F_n$ .

c. Let  $\frac{p}{q} = \frac{p_{j,n} + p_{j+1,n}}{q_{j,n} + q_{j+1,n}} \in Q_{n+1}$  and we have the continued fraction expansion

$$\frac{p}{q} = [0; a_1, \dots, a_{t-1}, a_t], \quad a_t \geq 2.$$

Find the continued fraction expansions for its "neighbours" in  $F_{n+1}$ , that is for the numbers  $\xi_{j,n}$  и  $\xi_{j+1,n}$ .

d. What is ordinary continued fraction algorithm from "Farey tree point of view"?

4. **Legendre Theorem.** Let

$$\frac{p}{q} = \frac{p_{j,n} + p_{j+1,n}}{q_{j,n} + q_{j+1,n}} \in Q_{n+1}, \quad \frac{p_-}{q_-} = \xi_{j,n} = \frac{p_{j,n}}{q_{j,n}}, \quad \frac{p_+}{q_+} = \xi_{j+1,n} = \frac{p_{j+1,n}}{q_{j+1,n}}$$

Consider the segment  $I = \left[ \frac{p_- + p}{q_- + q}, \frac{p + p_+}{q + q_+} \right]$ . Then the fraction  $\frac{p}{q}$  is a convergent fraction to  $\alpha$  if and only if  $\alpha \in I$ .

5. Monkemeyer's algorithm.

a. In  $\mathbb{R}^2$  we consider the triangle  $\Delta = A_0A_{-1}A_{-2}$  with vertices  $A_{-2} = (0, 1), A_{-1} = (1, 0), A_0 = (0, 0)$ . We deal with an inductive process. For a triangle  $A_\nu A_{\nu-1} A_{\nu-2}$  with vertices in rational points such that

$$A_{\nu-2} = \left( \frac{a_{\nu-2}}{c_{\nu-2}}, \frac{b_{\nu-2}}{c_{\nu-2}} \right), A_{\nu-1} = \left( \frac{a_{\nu-1}}{c_{\nu-1}}, \frac{b_{\nu-1}}{c_{\nu-1}} \right), \text{НОД}(a_j, b_j, c_j) = 1,$$

we consider its partition into two smaller triangles

$$A'_{\nu+1}A'_\nu A'_{\nu-1} = BA_\nu A_{\nu-1} \text{ и } A''_{\nu+1}A''_\nu A''_{\nu-1} = BA_\nu A_{\nu-2} \text{ где } B = \left( \frac{a_{\nu-1} + a_{\nu-2}}{c_{\nu-1} + c_{\nu-2}}, \frac{b_{\nu-1} + b_{\nu-2}}{c_{\nu-1} + c_{\nu-2}} \right)$$

Prove that the points which occur as the vertices of the triangles are just all rational points of the initial triangle  $\Delta$ .

(Suggestions:

- 1) Let  $\left( \frac{a_1}{b_1}, \dots, \frac{a_n}{b_n} \right) \in \mathbb{Q}^n$ . Prove that there exist unique  $A_1, \dots, A_n \in \mathbb{Z}, Q \in \mathbb{Z}_+$ , such that  $(A_1, \dots, A_n, Q) = 1$  and  $\frac{A_i}{Q} = \frac{a_i}{b_i}, \forall i$ .
- 2) Every three vectors

$$(a_j, b_j, c_j), j \in \{\nu - 1, \nu - 1, \nu\}$$

from a basis of  $\mathbb{Z}^3$ .

- 3) How one can calculate the area of  $\Delta = A_\nu A_{\nu-1} A_{\nu-2}$  in terms of  $c_\nu, c_{\nu-1}, c_{\nu-2}$ ?

### Exercises.

1. Hurwitz theorem revisited. Prove that for any  $\nu$  for the denominators of convergent fractions to  $\alpha$  one has either

$$q_{\nu-1} \|q_{\nu-1}\alpha\| < 1/\sqrt{5},$$

or

$$q_\nu \|q_\nu\alpha\| < 1/\sqrt{5},$$

or

$$q_{\nu+1} \|q_{\nu+1}\alpha\| < 1/\sqrt{5}.$$

2. What are the sets

$$\{\alpha \in [0, 1] : \alpha = [0; a_1, \dots, a_t], a_1 + \dots + a_t = n\}, \quad \{\alpha \in [0, 1] : \alpha = [0; a_1, \dots, a_t], a_1 + \dots + a_t \leq n\},$$

(here  $[0; a_1, \dots, a_t]$  is the continued fraction for  $\alpha$ )?

3. Farey map.

a. Consider the map on  $[0, 1]$  defined as

$$T(x) = \begin{cases} x/(1-x), & 0 \leq x \leq 1/2, \\ (1-x)/x, & 1/2 \leq x \leq 1. \end{cases}$$

What is  $T^{-n}(1)$  and  $T^{-n}(0)$ ?

- b. Find the sum of all the denominators of all numbers from  $T^{-n}(0)$ .

4. Legendre theorem simplified. Prove that if

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{2q^2} \quad \text{and} \quad (p, q) = 1$$

then  $\frac{p}{q}$  is a convergent fraction to  $\alpha$ .

5. Prove that in Monkemeyer's algorithm for any  $\alpha \in \Delta \setminus \mathbb{Q}^2$  there exists a sequence of nested triangles  $\Delta_\nu$  from the algorithm such that  $\{\alpha\} = \bigcap_{\nu=1}^{\infty} \Delta_\nu$ . (This property is called weak convergence of the algorithm.)