Farey tree (Lecture No. 4, 24.07.2024).

1. Hurwitz Theorem.
a. Mediant fraction. Consider two fractions $\frac{a}{b}$, $\frac{c}{d}$, $(a,b)=(c,d)=1$.
1) Show that the *mediant fraction* $\frac{a+c}{b+d}$ is located between $\frac{a}{b}$and $\frac{c}{d}$;
2) suppose that $\left|\frac{a}{b}-\frac{c}{d}\right|=\frac{1}{bd}$, then $\left|\frac{a}{b}-\frac{a+c}{b+d}\right|=\frac{1}{b(b+d)}$ and $\left|\frac{a+c}{b+d}-\frac{c}{d}\right|=\frac{1}{(b+d)d}$.
b. Hurwitz.
1) **Theorem 1.** *Suppose that* $\alpha\in\left[\frac{a}{b},\frac{c}{d}\right]$ *and* $\left|\frac{a}{b}-\frac{c}{d}\right|=\frac{1}{bd}$. *Then either* $\left|\alpha-\frac{a}{b}\right|\leqslant\frac{1}{\sqrt{5}b^2}$, *or* $\left|\alpha-\frac{c}{d}\right|\leqslant\frac{1}{\sqrt{5}d^2}$, *or* $\left|\alpha-\frac{a+c}{b+d}\right|\leqslant\frac{1}{\sqrt{5}(b+d)^2}$.
2) **Theorem 2.** *For any irrational* $\alpha\in\mathbb{R}$ *there exist infinitely many* $\frac{a}{q}$ *with* $\left|\alpha-\frac{a}{q}\right|<\frac{1}{\sqrt{5}q^2}$.
3) Why the constant $1/\sqrt{5}$ in Theorem 2 is optimal?

2. Basis of $\mathbb{Z}^2$. Consider two vectors $e_1=(a,b), e_2=(c,d)\in\mathbb{Z}^2$, with $ad-bc=\pm1$. Then
a. each of the vectors is primitive, that is $(a,b)=(c,d)=1$.
b. every integer point $z\in\mathbb{Z}^2$ can be written in a form $z=\lambda e_1+\mu e_2$ with integer λ,μ.

3. Stern-Brocot sequences. Define the sets F_n, $n=0,1,2,\ldots$ by the following inductive procedure. For $n=0$ put

$$F_0=\{0,1\}=\left\{\frac{0}{1},\frac{1}{1}\right\}.$$

If F_n is defined write it as a collection of increasing numbers:

$$0=\xi_{0,n}<\xi_{1,n}<\cdots<\xi_{N(n),n}=1, N(n),\quad \xi_{j,n}=\frac{p_{j,n}}{q_{j,n}},\quad (p_{j,n},q_{j,n})=1.$$

Then we define F_{n+1} as

$$F_{n+1}=F_n\cup Q_{n+1}$$

where

$$Q_{n+1}=\left\{\frac{p_{j,n}+p_{j+1,n}}{q_{j,n}+q_{j+1,n}},\quad j=0,\ldots,N(n)-1\right\}.$$

a. What is $N(n)$?
b. Prove that for every rational $\xi\in[0,1]$ there exists n such that $\xi\in F_n$.
c. Let $\frac{p}{q}=\frac{p_{j,n}+p_{j+1,n}}{q_{j,n}+q_{j+1,n}}\in Q_{n+1}$ and we have the continued fraction expansion

$$\frac{p}{q}=[0;a_1,...,a_{t-1},a_t],\quad a_t\geqslant 2.$$

Find the continued fraction expansions for its "neighbours" in F_{n+1}, that is for the numbers $\xi_{j,n}$ и $\xi_{j+1,n}$.
d. What is ordinary continued fraction algorithm from "Farey tree point of view"?

Exercises.

1. Hurwitz theorem revisited. Prove that for any ν for the deniminators of convergent fractions to α one has either

$$q_{\nu-1}||q_{\nu-1}\alpha||<1/\sqrt{5},$$

or

$$q_\nu||q_\nu\alpha||<1/\sqrt{5},$$

or

$$q_{\nu+1}||q_{\nu+1}\alpha|| < 1/\sqrt{5}.$$

2. What are the sets

$$\{\alpha \in [0,1]: \ \alpha = [0; a_1, ..., a_t], \ \ a_1 + ... + a_t = n\}, \quad \{\alpha \in [0,1]: \ \alpha = [0; a_1, ..., a_t], \ \ a_1 + ... + a_t \leqslant n\},$$

(here $[0; a_1, ..., a_t]$ is the continued fraction for α)?

3. Farey map.
a. Concider the map on $[0, 1]$ defined as

$$T(x) = \begin{cases} x/(1-x), & 0 \leqslant x \leqslant 1/2, \\ (1-x)/x, & 1/2 \leqslant x \leqslant 1. \end{cases}$$

What is $T^{-n}(1)$ and $T^{-n}(0)$?
b. Find the sum of all the denominators of all numbers from $T^{-n}(0)$.

4. **Legendre Theorem.**

a. *Let*
$$\frac{p}{q} = \frac{p_{j,n} + p_{j+1,n}}{q_{j,n} + q_{j+1,n}} \in Q_{n+1}, \quad \frac{p_-}{q_-} = \xi_{j,n} = \frac{p_{j,n}}{q_{j,n}}, \quad \frac{p_+}{q_+} = \xi_{j+1,n} = \frac{p_{j+1,n}}{q_{j+1,n}}$$
Consider the segment $I = \left[\frac{p_-+p}{q_-+q}, \frac{p+p_+}{q+q_+}\right]$. *Then the fraction* $\frac{p}{q}$ *is a convergent fraction to* α *if and only if* $\alpha \in I$.

b. Legendre theorem simplified. Prove that if

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2} \quad \text{and} \quad (p, q) = 1$$

then $\frac{p}{q}$ is a convergent fraction to α.

5. Monkemeyer's algorithm.

a. In $\mathbb{R}^2$ we consider the triangle $\Delta = A_0A_{-1}A_{-2}$ with vertices $A_{-2} = (0, 1), A_{-1} = (1, 0), A_0 = (0, 0)$. We deal with an inductive process. For a triangle $A_\nu A_{\nu-1}A_{\nu-2}$ with verices in rational points such that
$$A_{\nu-2} = \left(\frac{a_{\nu-2}}{c_{\nu-2}}, \frac{b_{\nu-2}}{c_{\nu-2}}\right), A_{\nu-1} = \left(\frac{a_{\nu-1}}{c_{\nu-1}}, \frac{b_{\nu-1}}{c_{\nu-1}}\right), \ \text{g.c.d.}\,(a_j, b_j, c_j) = 1,$$
we consider its partition into two smaller triangles

$$A'_{\nu+1}A'_\nu A'_{\nu-1} = BA_\nu A_{\nu-1} \ \text{ and } \ A''_{\nu+1}A''_\nu A''_{\nu-1} = BA_\nu A_{\nu-2} \ \text{ where } \ B = \left(\frac{a_{\nu-1} + a_{\nu-2}}{c_{\nu-1} + c_{\nu-2}}, \frac{b_{\nu-1} + b_{\nu-2}}{c_{\nu-1} + c_{\nu-2}}.\right)$$

Prove that the points which occur as the vertices of the triangles are just all rational points of the initial triangle Δ.
(Suggestions:
1) Let $\left(\frac{a_1}{b_1}, ..., \frac{a_n}{b_n}\right) \in \mathbb{Q}^n$. Prove that there exist unique $A_1, ..., A_n \in \mathbb{Z}, Q \in \mathbb{Z}_+$, such that $(A_1, ..., A_n, Q) = 1$ and $\frac{A_i}{Q} = \frac{a_i}{b_i}, \forall\, i$.
2) Every three vectors
$$(a_j, b_j, c_j), j \in \{\nu - 1, \nu - 1, \nu\}$$

form a basis of $\mathbb{Z}^3$.
3) How one can calculate the area of $\Delta = A_\nu A_{\nu-1} A_{\nu-2}$ in terms of $c_\nu, c_{\nu-1}, c_{\nu-2}$?

b. Prove that in Monkemeyer's algorithm for any $\alpha \in \Delta \backslash \mathbb{Q}^2$ there exists a sequence of nested triangles Δ_ν from the algorithm such that $\{\alpha\} = \bigcap_{\nu=1}^{\infty} \Delta_\nu$. (This property is called weak convergence of the algorithm.)