Basic Number Theory: Euler, Möbius and Farey. Lecture 5. 25.07.2024

0. Basic sums.

$$\sum_{n=1}^{N}\frac{1}{n}=\log N+\gamma+\frac{\theta_N}{N},\ \text{ where }\ \theta_N \text{ is bounded;}\ \ \sum_{p\le x}\frac{1}{p}\to\infty, x\to\infty;\ \ \sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6}.$$

1. Let us recall the definitions of
a. Euler Function

$$\varphi(n)=\#\{x\in\mathbb{Z}:\ 1\le x\le n,\ (x,n)=1\};$$

b. Möbius function

$$\mu(n)=\begin{cases}1, & n=1,\\ 0, & p \text{ prime},\ p^2|n,\\ (-1)^r, & n=p_1\cdots p_r \ \ - \text{ different primes.}\end{cases}$$

c. Number of divisors function $\tau(n)=\sum_{d|n}1$.
d. Sum of divisors function $\sigma(n)=\sum_{d|n}d$.
These functions are multiplicative.
A function $f(x), x\in\mathbb{Z}_+$ is *multiplicative* if

$$f(nm)=f(n)f(m)\ \text{ for all }\ n,m\in\mathbb{Z}_+\ \text{ under the condition }\ (n,m)=1.$$

2. Most important formulas:

$$\sum_{d:\,d|n}\mu(d)=\begin{cases}1, & n=1,\\ 0, & n>1.\end{cases}\qquad \varphi(n)=n\cdot\sum_{d:\,d|n}\frac{\mu(d)}{d},\qquad \sum_{d:\,d|n}\varphi(d)=n.$$

3. What are sums $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}$?

4. Consider the sum

$$g_N(x)=\sum_{q:\,q\le N}\ \sum_{1\le a\le xq}\ \sum_{d:\,d|\text{g.c.d.}(a,q)}\mu(d).$$

Prove that

$$g_N(1)=\sum_{q\le N}\varphi(q).$$

What is the meaning of this sum when $x\in[0,1]$?

5. Find limits

$$\lim_{N\to\infty}\frac{g_N(1)}{N^2},\quad \lim_{N\to\infty}\frac{g_N(x)}{N^2}\ \text{ and }\ \lim_{N\to\infty}\frac{g_N(x)}{g_N(1)}.$$

6. Consider *Farey series*

$$\mathcal{F}_N=\left\{\frac{p}{q}\in[0,1]:\quad q\le N\right\}.$$

a. Now many numbers are there in $\mathcal{F}_N$?

b. Show that $g_N(x) = \#\{\xi \in \mathcal{F}_N : \ \xi \le x\}$ and calculate the limit

$$\lim_{N\to\infty} \frac{g_N(x)}{\#\mathcal{F}_N}, \qquad x \in [0,1].$$

7. Other mean values: calculate limits

$$a. \lim_{x\to\infty} \frac{\sum_{n\le x} \varphi(n)}{x^2}; \quad b. \lim_{x\to\infty} \frac{\sum_{n\le x} \tau(n)}{x\log x}; \quad c. \lim_{x\to\infty} \frac{\sum_{n\le x} \sigma(n)}{x^2}; \quad d. \lim_{x\to\infty} \frac{\sum_{n\le x} \mu(n)}{x};$$

Exercises.

1. Möbius inversion formula. Let $g(x)$ and $G(x)$ be two functions of $x \in \mathbb{Z}_+$. Then

$$G(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} \mu(d) G\left(\frac{n}{d}\right).$$

2. Let $\tau(n) = \sum_{d|n} 1$ be the number of divisors of n and $\sigma(n) = \sum_{d|n} d$ be the sum of divisors of n. Calculate sums

a. $\sum_{d|n} \mu(d)\tau(n/d)$;

b. $\sum_{d|n} \mu(d)\sigma(n/d)$.

3. Dirichlet series. Consider infinite sums

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^2} \text{ and } \sum_{m=1}^{\infty} \frac{g(m)}{m^2}.$$

Let us write the product in a form

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^2}\right) \cdot \left(\sum_{m=1}^{\infty} \frac{g(m)}{m^2}\right) = \sum_{k=1}^{\infty} \frac{h(k)}{k^2}.$$

What is $h(\cdot)$? (We suppose that we can change the order of INFINITE summation everywhere, for example this is possible when the series are absolutely convergent.)

4. Calculate sum $\sum_{n\le x} \mu(n) \left[\frac{x}{n}\right]$.

5. Calculate the limits from 7. which were not calculated during the lecture.

6. Prove that

   a. $\forall\, \varepsilon > 0 \quad \lim_{n\to\infty} \frac{\tau(n)}{n^\varepsilon} = 0$.

   b. $\forall\, A > 0 \quad \limsup_{n\to\infty} \frac{\tau(n)}{(\log n)^A} = \infty$.

7. Find all limit points of the sequence $\frac{\varphi(n)}{n}$.

   Simplification: does there exist a sequence n_k such that $\lim_{k\to\infty} \frac{\varphi(n_k)}{n_k} = 0$?

8. For two functions $f(n), g(n), n \in \mathbb{Z}_+$ define the operation

$$(f \circ g)(n) = \sum_{d|n} f(d) g(n/d).$$

This operation is called *Dirichlet convolution*.
8.1. Prove that it is commutative (that is $f \circ g = g \circ f$) and associative (that is $f \circ (g \circ h) = (f \circ g) \circ h$).
8.2. Write formulas from 1.2. Möbius inversion formula, formulas from exercises 2 and 3 in terms of $\circ$.
Suggestion: you should use functions

$$I(x) = 1 \ \forall x, \ \ E(x) = x \ \forall\, x, \ \ J(x) = \begin{cases} 1, & x = 1, \\ 0, & x > 1. \end{cases}$$