Minkowski Question Mark function (Lecture 6. 26.07.2024).

For the Stern-Brocot sequences F_n we consider the function

$$?(x) = \lim_{n\to\infty} \frac{\#\{\xi \in F_n : \ \xi \le x\}}{\#F_n}, \quad x \in [0,1], \tag{1}$$

which is called the Minkowski Question Mark function. By the way, a-pripori it is not clear why this limit exists. So the next statements 1, 2, 3, 4 ensure that it is really so.

1. For elements $\xi_{j,n}$ of the sequence F_n one has $?(\xi_{j,n}) = \frac{j}{2^n}$.

2. The function $?(x)$ increases monotonically for rational values of x.

3. For irrational $x \in [0,1]$ one has

$$\sup_{\mathbb{Q}\ni\frac{p}{q}<x} ?\left(\frac{p}{q}\right) \le \inf_{\mathbb{Q}\ni\frac{p}{q}>x} ?\left(\frac{p}{q}\right).$$

4. The limit in (1) exists.

5. $?(x)$ is continuous in every point.

6. **Lebesgue Theorem.** *Every monotone function on an interval I has derivative in almost every (in the sense of Lebesgue measure) point of I.*
We will not prove this theorem. A. Lebesgue thought this theorem to be the best of his results. In particular, from this theorem it follows that $?(x)$ has derivative $?'(x)$ almost everywhere.

7. **Theorem 1.** *If at point $x \in [0,1]$ the derivative $?'(x)$ exists then either $?'(x) = 0$ or $?'(x) = +\infty$.*

8. What is $?'(x)$ for $x \in \mathbb{Q}$?

9. Stable points. The equation $?(x) = x$ has at least 5 solutions.

10. **Theorem 2.** *Let all the partial quotients in the continued fraction expansion of irrational x are ≤ 4. Then $?'(x) = +\infty$.*

11. **Koksma's inequality.**

$$\left| \int_0^1 f(x)dx - \frac{1}{q}\sum_{j=1}^{q} f(\xi_j) \right| \le \frac{V[f]D(\Xi)}{q},$$

where $D(\Xi)$ is the discrepancy of the sequence $\Xi = (\xi_1, ..., \xi_q)$ and

$$V[f] = \sup_{x_1<x_2<...<x_t} \sum_{j=0}^{t-1} |f(x_{j+1}) - f(x_j)|$$

is the full variation of f.

12. **Theorem 3.**

$$\left| 2^n \cdot \int_0^1 ((?(x) - x)^2 dx - \sum_{j=0}^{2^n} \left(\xi_{j,n} - \frac{j}{2^n}\right)^2 \right| \le 4.$$

Exercises.

1. Prove that if $\frac{a}{b}, \frac{c}{d} \in [0,1]$ and $ad - bc = 1$, $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$ then

$$?\left(\frac{a}{b} \oplus \frac{c}{d}\right) = \frac{1}{2}\left(?\left(\frac{a}{b}\right) + ?\left(\frac{c}{d}\right)\right).$$

2. a. For irrational $x = [0; a_1, a_2, ..., a_n, ...]$ prove the formula

$$?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + ... + \frac{(-1)^{n+1}}{2^{a_1+...+a_n-1}} + ...$$

b. What is he analogous formula for *rational* $x = [0; a_1, a_2, ..., a_n]$?

3. Prove that if $x \in [0,1]$ is a quadratic irrationality then $?(x) \in \mathbb{Q}$.

4. Construct a point x such that $?'(x)$ does not exist (not equal neither to 0 nor to $+\infty$).

5. Prove Theorem 2.

Three simplifications of this problem.

a. Prove that $?'\left(\frac{\sqrt{5}-1}{2}\right) = +\infty$.

b. Prove that $?'(x) = +\infty$ for all irrational x with partial quotients ≤ 2.

c. Prove that $?'(x) = +\infty$ for all irrational x with partial quotients ≤ 3.

6. Find $?'(x_d)$ for every $x_d = [0; \overline{d}]$.

7. Prove that the equation $?(x) = x$ an irrational solution.

8. a. Assume that $?([0; a_1, ..., a_{n-1}]) = [0; b_1, ..., b_k]$ and $a_1 + ... + a_{n-1} = s + 1 > 2$. Prove $b_1 + ... + b_k > s + 1$.

b. Prove that the equation $?(x) = x$ has exactly three rational solutions.

9. Suppose that there exists q_0 such that for all $q \geq q_0$ we have

$$\left|?\left(\frac{p}{q}\right) - \frac{p}{q}\right| > \frac{1}{2q^2}.$$

Then equation $?(x) = x$ has just 5 solutions.

10. Let $f(x)$ be function inverse to $?(x)$. Prove

$$\int_0^1 (f(x) - x)^2 dx = \int_0^1 (?(x) - x)^2 dx.$$