Uniform Distribution Theory (Lecture 7, 29 July 2024).

1. **Definition 1.** An infinite sequence ξ_j, $j = 1, 2, 3, ...$ of points from the unit interval $[0, 1)$ is called uniformly distributed (U.D.) if the following holds. For every $\gamma \in [0, 1)$ and for every $q \in \mathbb{Z}_+$ the quantity

$$N_q(\gamma) = |\{n \in \mathbb{Z}_+ : \ n \leq q, \{\xi_n\} \leq \gamma\}|$$

satisfies

$$\lim_{q\to\infty} \frac{N_q(\gamma)}{q} = \gamma,$$

or

$$N_q(\gamma) = \gamma q + o(q), \ \ q \to \infty.$$

2. **Definition 2.** Consider a finite sequence $\Xi = \{\xi_1, ..., \xi_q\}$. *Discrepancy* of this sequence is defined as

$$D(\Xi) = \sup_{\gamma\in[0,1)} |\frac{N_q(\gamma)}{q} - \gamma|.$$

3. What can you say about the discrepancy of the sequence
a. $\frac{0}{q}, \frac{1}{q}, \frac{2}{q}, ..., \frac{q-1}{q}$?
b. $\frac{a\cdot 0}{q}, \frac{a\cdot 1}{q}, \frac{a\cdot 2}{q}, ..., \frac{a\cdot(q-1)}{q}$, where $(a, q) = 1$?

4. Obvious proposition. For an infinite sequence ξ_j, $j = 1, 2, 3, ...$ consider its beginning $\Xi_q = \{\xi_1, ..., \xi_q\}$. Then if

$$\lim_{q\to\infty} D_q = 0, \ \ D_q = D(\Xi_q),$$

then the sequence is U.D.

5. Example. For the sequence $\xi_n = \{\sqrt{n}\}$ we have the bound $D_q = O(\frac{1}{\sqrt{q}})$.

6. **Ostrowski's theorem.** *Let* $a = [a_0; a_1, ..., a_\nu, ...]$ *be irrational number and there exists* M *such that all the partial quotients in its continued fraction are bounded by* M*:*

$$a_j \leq M, \ \ \forall j$$

Then the discrepancy of the sequence $\{\alpha n\}, n = 1, 2, 3, ...$ *satisfy*

$$D_q \leq 100M\frac{\log q}{q}, \ \ \forall q.$$

To prove this theorem we need Ostrowski's numerical system. Let q_ν be the sequence of the denominators of convergent fractions to $\alpha = [0; a_1, a_2, ...]$. Then every positive integer q can be written in a form

$$q = b_0q_0 + b_1q_1 + b_2q_2 + ... + b_tq_t, \ \ b_j \leq a_{j+1}.$$

7. **Weyl Criteria.**
a. **Theorem 1.** *The sequence* ξ_j, $j = 1, 2, 3, ...$ *is U.D. if and only if for any continuous function* $f : [0, 1] \to \mathbb{R}(\mathbb{C})$ *one has*

$$\lim_{q\to\infty} \frac{1}{q}\sum_{j=1}^{q} f(\xi_j) = \int_0^1 f(x)dx.$$

b. **Theorem 2.** *The sequence* ξ_j, $j = 1, 2, 3, ...$ *is U.D. if and only if for any* $m \in \mathbb{Z} \setminus \{0\}$ *one has*

$$\lim_{q \to \infty} \frac{1}{q} \sum_{j=1}^{q} e^{2\pi i m \xi_j} = 0, \quad e^{ix} = \cos x + i \sin x.$$

We will not prove the second criteria as we do not know Weierstrass theorem that any periodic continuous function on $[0, 1]$ can be approximated by a trigonometric polynomial, that is

$$\forall\, f(x) \text{ continuous and periodic on } [0,1] \forall\, \varepsilon$$

$$\exists\, P(y) - \text{ polynomial, such that } \sup_{x \in [0,1]} |f(x) - P(e^{2\pi i x})| \le \varepsilon.$$

8. **Roth-Schmidt theorem.** *There exists an absolute constant* c *such that for any infinite sequence*

$$\limsup_{n \to \infty} \frac{q \cdot D_q}{\log q} > c.$$

Exercises.

0. Prove that for any set Ξ of q elements in $[0, 1)$ we have $D(\Xi) \ge \frac{1}{2(q+1)}$.
1. Obtain upper bound for the discrepancy of the sequence $\xi_n = \{\sqrt[3]{n}\}$.

2 a. Is $\xi_n = \{\log n\}$ U.D. or not?
b. For which β the sequence $\xi_n = \{(\log n)^\beta\}$ is U.D.?

3. Inverse to 4 from the lecture. Prove that if the sequence is U.D. Then $D_q = o(q)$, $q \to \infty$.

4. Van der Corput sequence. For $n \ge 1$ consider the dyadic expansion

$$n - 1 = \sum_{j=0}^{s} a_j 2^j.$$

now we define

$$\xi_n = \sum_{j=0}^{s} \frac{a_j}{2^{j+1}}.$$

Prove that for this sequence $D_q = O(\log q)$.

5. Roth-Schmidt's theorem for exponential function. Prove that for any α for the discrepancy D_q of the sequence $\{\alpha 2^n\}_{n=1}^{q}$ one has

$$\limsup_{q \to \infty} \frac{q \cdot D_q}{\log q} > 0.$$

6.
a. Let $\Xi : 0 < x_1 < x_2 < ... < x_N < 1$, $\Xi \subset [0, 1)$. Prove that

$$D(\Xi) = \max_{1 \le i \le N} \max \left(\left| x_i - \frac{i}{n} \right|, \left| x_i - \frac{i-1}{N} \right| \right).$$

The same is true under a weaker condition $0 \leq x_1 \leq x_2 \leq ... \leq x_N < 1$,
b. Prove that if

$$V[f] = \sup_{x_1 < x_2 < ... < x_t} \sum_{j=0}^{t-1} |f(x_{j+1}) - f(x_j)| < \infty,$$

then

$$\frac{1}{N} \sum_{n=1}^{N} f(x_v) - \int_0^1 f(t)dt = \sum_{n=0}^{N} \int_{x_n}^{x_{n+1}} \left(t - \frac{n}{N}\right) df(x),$$

here

$$\int_a^b g(x)df(x) = \lim_{\eta_i \in [\xi_{i-1}, \xi_i], \max_i |\xi_i - \xi_{i-1}| \to 0} g(\eta_i)(f(\xi_i) - f(\xi_{i-1}))$$

is Stieltjes integral. (We believe that Stieltjes integral exists when $g(x)$ is continuous and $f(x)$ has bounded variation $V[f]$.)
c. From the previous lecture: **Koksma's inequality.** Prove

$$\left| \int_0^1 f(x)dx - \frac{1}{q} \sum_{j=1}^{q} f(\xi_j) \right| \leq \frac{V[f]D(\Xi)}{q},$$

where $D(\Xi)$ is the discrepancy of the sequence $\Xi = (\xi_1, ..., \xi_q)$ and

$$V[f] = \sup_{x_1 < x_2 < ... < x_t} \sum_{j=0}^{t-1} |f(x_{j+1}) - f(x_j)|$$

is the full variation of f.

7. Construct α such that the sequence $\{\alpha n!\}$ is U.D.