Rational subspaces and Linear Diophantine equation (Lecture 11, 2 August 2024)

0. How to solve in integers x_1, x_2 linear equation $a_1x_1 + a_2x_2 = b$ (here $a_1, a_2, b \in \mathbb{Z}$)?

1. Algorithm of solving equation

$$a_1x_1 + ... + a_nx_n = b, \quad a_1, ..., a_n, b \in \mathbb{Z}.$$

a) Write matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix};$$

b) we allowed to use the following elementary procedure: to one of the columns we can add another one or from one of the colimns we can subtract another one; by means of this procedure we should transfer matrix A to matrix

$$C = \begin{pmatrix} 0 & \cdots & 0 & d & 0 & \cdots & 0 \\ c_{1,1} & \cdots & c_{1,k-1} & c_{1,k} & c_{1,k+1} & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n,1} & \cdots & c_{n,k-1} & c_{n,k} & c_{n,k+1} & \cdots & c_{n,n} \end{pmatrix};$$

c) conclusion: if $d \not| b$, then the equation has no solutions; if $d|b$, then all the solutions of the equation can be found by formula

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = t_1 \begin{pmatrix} c_{1,1} \\ \vdots \\ c_{n,1} \end{pmatrix} + \cdots + t_{k-1} \begin{pmatrix} c_{1,k-1} \\ \vdots \\ c_{n,k-1} \end{pmatrix} + \frac{b}{d} \begin{pmatrix} c_{1,k} \\ \vdots \\ c_{n,k} \end{pmatrix} + t_{k+1} \begin{pmatrix} c_{1,k+1} \\ \vdots \\ c_{n,k+1} \end{pmatrix} + \cdots + t_n \begin{pmatrix} c_{1,n} \\ \vdots \\ c_{n,n} \end{pmatrix},$$

where $t_1, ..., t_{k-1}, t_{k+1}, ..., t_n$ are $(n-1)$ arbitrary integer parameters.

2. Let $0 \leq k \leq n$. In the lecture we will give definitions of:
   a) lattice;
   b) linear rational subspace of dimension k in $\mathbb{R}^n$;
   c) affine rational subspace of dimension k in $\mathbb{R}^n$;
   d) height $H(\mathcal{A})$ of rational (or affine) subspace $\mathcal{A}$ of dimension k in $\mathbb{R}^n$;
   e) linear totally irrational subspace of dimension k in $\mathbb{R}^n$;
   f) angle between two subspaces $\mathcal{A}$ and $\mathcal{B}$.
   Try to understand these definitions.

3. Let $a_1, ..., a_n, b \in \mathbb{Z}$, g.c.d$(a_1, ..., a_n) = 1$. Why

$$\mathcal{A} = \{(x_1, .., x_n) \in \mathbb{R}^n : a_1x_1 + ... + a_nx_n = b\}$$

is a $(n-1)$-dimensional rational subspace in $\mathbb{R}^n$ and what is its height $H(\mathcal{A})$?

4. **Theorem** (W.M. Schmidt). Let $\mathcal{B}$ be a linear two-dimensional totally irrational subspace in $\mathbb{R}^4$. Then there exist infinitely many linear two-dimensional rational subspaces $\mathcal{B} \subset \mathbb{R}^4$ such that

$$\text{angle between } \mathcal{A} \text{ and } \mathcal{B} \leq \text{ const } (H(B))^{-3}.$$

5. **Terrible question.** Why totally irrational subspaces exist?

Exercises 2 August 2024

(exercises from previous days which were not solved)

From Lecture No. 1:

1. Is Liouville's theorem valid for complex algebraic numbers?

2. Prove that the number $\sum_{n=0}^{\infty} \frac{1}{2^{2^{n^2}}}$ is not algebraic.

From Lecture No. 2:

3. Prove that for any $q \in \mathbb{Z}_+$ there exists $a \in \mathbb{Z}$ such that $(a, q) = 1$ and in the decomposition

$$b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cdots - \cfrac{1}{b_\nu}}}$$

we have $b_j \leq 5 \, \forall j$.

From Lecture No. 3:

4. Consider rational number $\frac{a}{b}, (a, b) = 1$. Find out how many are there representations of $\frac{a}{b}$ in a form

$$\frac{a}{b} = a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_n + \cdots + \cfrac{\varepsilon_k}{a_k}}},$$

with the following conditions:
1) $a_j \in \mathbb{N}, a_k \geq 2$;
2) $\varepsilon_j \in \{-1, +1\}$;
3) if $a_j = 1$ then $\varepsilon_{j+1} = +1$.

From Lecture No. 6:

5. Prove that $?'(x) = +\infty$ for all irrational x with partial quotients ≤ 2.

From Lecture No. 8:

6. Roth-Schmidt's theorem for exponential function. Prove that for any α for the discrepancy D_q of the sequence $\{\alpha 2^n\}_{n=1}^q$ one has

$$\limsup_{q \to \infty} \frac{q \cdot D_q}{\log q} > 0.$$

7. Construct α such that the sequence $\{\alpha n!\}$ is U.D.

From Lecture No. 9:

8. Prove that there exist a number α of the form $\alpha = [0; a_1, ..., a_n, ...] : \ a_j \leq 4$, which can be written in the base 3 without digit 1.

9. Consider the set $W_k = \{\alpha \in [0,1] : \text{ all partial quotients of } \alpha \text{ are } \geq k\}$. Prove that $W_2 + W_2 = [0,1]$.

10. Prove that if in continued fraction expansion of irrational α there exist infinitely many partial quotients $a_n \geq 2$, then

a) $\lambda(\alpha) \leq \frac{1}{\sqrt{8}} = \liminf_{t\to\infty} t \cdot \psi_\alpha(t) = \lambda(\sqrt{2})$, b) $d(\alpha) = \limsup_{t\to\infty} t \cdot \psi_\alpha(t) \geq d\left(\frac{1+\sqrt{3}}{2}\right)$.

11. Ray in Dirichlet spectrum. Prove that there exists $d^* < 1$ such that

$$[d^*, 1] \subset \mathbb{D} = \{d \in \mathbb{R} : \ \exists\, \alpha \text{ such that } \ d = \limsup_{t\to\infty} t \cdot \psi_\alpha(t)\}.$$

One new exercise for today's lecture:

12. Let $a_{i,j}, b_j$ be integers. Consider the system

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 = b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 = b_2. \end{cases}$$

Prove that this system has an integer solution (x_1, x_2, x_3) if and only if

$$\text{g.c.d.}\left(\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}, \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix}, \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} \right) =$$

$$= \text{g.c.d.}\left(\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}, \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix}, \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix}, \begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix}, \begin{vmatrix} a_{1,2} & b_1 \\ a_{2,2} & b_2 \end{vmatrix}, \begin{vmatrix} a_{1,3} & b_1 \\ a_{2,3} & b_2 \end{vmatrix} \right).$$