Lagrange spectrum and Hall's Ray (Lecture 9, 31 July 2024).

For a real α consider irrationality measure function

$$\psi_\alpha(t) = \min_{1\le x\le t,\, x\in\mathbb{Z}} ||x\alpha||.$$

Largange Theorem. $\psi_\alpha(t) = ||q_\nu\alpha||$ for $q_\nu \le t < q_{\nu+1}$.

Lagrange constant

$$\lambda(\alpha) = \liminf_{t\to+\infty} t\cdot\psi_\alpha(t),$$

and Dirichlet constant

$$d(\alpha) = \limsup_{t\to+\infty} t\cdot\psi_\alpha(t).$$

What is the maximal and minimal possible values of value of $\lambda(\alpha)$ and $d(\alpha)$?

The Lagrange spectrum $\mathbb{L}$ is defined as

$$\mathbb{L} = \{\lambda\in\mathbb{R}\setminus\mathbb{Q} : \text{ there exists } \alpha\in\mathbb{R} \text{ such that } \lambda = \lambda(\alpha)\}.$$

The Dirichlet spectrum $\mathbb{D}$ is defined as

$$\mathbb{D} = \{d\in\mathbb{R} : \text{ there exists } \alpha\in\mathbb{R}\setminus\mathbb{Q} \text{ such that } d = d(\alpha)\}.$$

Theorem. $\exists\,\lambda^* > 0 : \quad [0,\lambda^*]\subset\mathbb{L}$.

This theorem follows from the

Main Lemma. *Suppose that*

$$K_r = \{\alpha\in[0,1]\setminus\mathbb{Q};\ \alpha = [0; a_1, ..., a_n, ...] : \ a_j \le r\ \forall j\}.$$

Then $K_4 + K_4$ is a segment.

Plan of the proof of Main Lemma.

A. What is the definition of τ-thick set

$$K = I \setminus \left(\bigcup_j \Delta_j\right),$$

where I is a segment and $\Delta_j\subset I$ are subintervals of I which do not intersect each other?
B. If K is a 1-thick set, than $K+K$ is a segment.
C. K_4 is an 1-thick set.

Exercises.

1. Prove that if

$$\alpha = [0; a_1, ..., a_n, x], \quad \beta = [0; a_1, ..., a_n, y]$$

then

$$|\alpha-\beta| = \frac{|x-y|}{(xq_n+q_{n-1})(yq_n+q_{n-1})}.$$

2. a. Prove that all the numbers form $[0,1]$ can be written which can be written as a sum of two numbers: the first summand can be written in the base 3 without digit 1 and simultaneously the second summand can be written in the base 5 without digit 2.

b. Prove that there exist a number α of the form $\alpha = [0; a_1, ..., a_n, ...] : \ a_j \le 4$, which can be written in the base 3 without digit 1.

3. Suppose that $\tau_1 \cdot \tau_2 \ge 1$. Prove that any two τ_1-thick set and τ_2-thick set have non-empty intersection provided that one does not belong to a connected part of the complement to another one.

4. How thick is a. K_2? b. K_3?

5. Consider the set $W_k = \{\alpha \in [0,1] : \text{all partial quotients of } \alpha \text{ are } \ge k\}$.
a. Prove that $W_2 + W_2 = [0,1]$.
b. Prove that $W_k + ... + W_k = [0,1]$ (k summands).

6. Prove that $K_3 + K_3 + K_3$ is a segment.

7. How many copies of K_2 one should take to have the sum $K_2 + ... + K_2$ to be a segment?

8. Prove that if a set is a τ-thick set we can throw away intervals Δ_j from I in such a way, that their lengths are decreasing. (This will preserve τ-thickness property.)

9. Prove that if in continued fraction expansion of irrational α there exist infinitely many partial quotients $a_n \ge 2$, then
a) $\lambda(\alpha) \le \frac{1}{\sqrt{8}} = \lambda(\sqrt{2})$, b) $d(\alpha) \ge d\left(\frac{1+\sqrt{3}}{2}\right)$.

11. Ray in Dirichlet spectrum. Prove that there exists $d^* < 1$ such that

$$[d^*, 1] \subset \mathbb{D} = \{d \in \mathbb{R} : \ \exists\, \alpha \text{ such that } \ d = \limsup_{t\to\infty} t \cdot \psi_\alpha(t)\}.$$