PROJECTS FOR MATHCAMP IN GEOMETRY AND TOPOLOGY

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ABSTRACT. Here are two proposed projects for Geometry and Topology. In the first paragraph of each project, we give a motivation for the study to make it more interesting. On the other hand, don't get scared if you do not understand the motivation: you can finish the project with your own motivation. Also, just like any other project, you may just finish a certain part of it. Work out and elaborate mathematically as much as you can.

Both questions use critical points and gradient flow of a smooth function. Question 1 uses unstable submanifolds to decompose a manifold. I will introduce these in the coming few classes.

(1) Symmetries on a space are crucial features in mathematics and physics. We consider functions (or physical observables) that are invariant under symmetries. This means we consider functions that descend to the 'space quotient out by symmetries'. Unfortunately, the quotient space is often singular. The space $\mathbb{RP}^\infty$ studied below is an example of a class of spaces that is useful to resolve such singularities.

The goal is to compute the homology of $\mathbb{RP}^\infty$ by studying critical points and gradient flow of a smooth function defined on it.

(a) First consider $\mathbb{RP}^2$, which is the set of all ratios $[x : y : z]$ of three real numbers (where $(x, y, z) \neq (0, 0, 0)$). Find a manifold structure on $\mathbb{RP}^2$.

(b) Fix numbers $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Consider the function

$$\frac{\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2}{x^2 + y^2 + z^2}$$

on $\mathbb{R}^3$. Show that it descends to a smooth function f on $\mathbb{RP}^2$.

(c) Find the critical points of f. Compute the Hessian matrices (second derivatives) at these critical points and their indices (numbers of negative eigenvalues).

(d) Consider the map $\mathbb{S}^2 \to \mathbb{RP}^2$, where $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$. Show that this is a double cover, and the standard metric of $\mathbb{S}^2$ descends to a well-defined metric on $\mathbb{RP}^2$.

(e) Solve the differential equation for gradient flow trajectories:

$$\gamma'(t) = -\mathrm{grad}(f)(\gamma(t))$$

on $\mathbb{RP}^2$, for a given initial condition $\gamma(0) = [x_0 : y_0 : z_0]$ for some fixed $[x_0 : y_0 : z_0] \in \mathbb{RP}^2$. (Hint: note that $f|_{\mathbb{S}^2} = g|_{\mathbb{S}^2}$, where $g = \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$. Compare the trajectories of f and g on $\mathbb{RP}^2$. Also, compare with the trajectories of g on $\mathbb{R}^3 - \{0\}$.)

(f) Find the unstable submanifolds of each critical point of f. Thus, compute the complex formed by the unstable submanifolds and its homology.
(g) Generalize the above to $\mathbb{RP}^n$ (the set of all ratios $[x_0 : \ldots : x_n]$ where $(x_0, \ldots, x_n) \neq \vec{0}$) for $n > 2$.
(h) What do you think if we take $n \to \infty$? How would you define $\mathbb{RP}^\infty$? How about $\mathbb{S}^\infty$? Any relation between them?

(2) Minimization problem is a core topic of Calculus and has vast applications in all branches of science. The following problem reveals the mathematical nature of AI, which composes linear maps with non-linear functions in a smart way to universally approximate any given function.

Consider a collection of hyperplanes in $\mathbb{R}^n$:

$$H_i = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : l_i(x_1, \ldots, x_n) = 0\}$$

for $i = 1, \ldots, N$. Suppose there is a real-valued function f defined on the complement

$$H^c := \mathbb{R}^n - \bigcup_{i=1}^{N} H_i$$

such that f is constant in each connected component. The goal is to find a strategy of using Calculus to minimize f.

(a) Give a mathematical description of each component C of H^c.
(b) Consider the function

$$\frac{1}{1 + T^{-x}}$$

for $x \in \mathbb{R}$, $T > 0$. For each $x \in \mathbb{R}$, find its series expansion at $T = 0$ and the domain of convergence. Then regard it as a family of functions in $x \in \mathbb{R}$ over $T > 0$, and find the pointwise limit as $T \to 0$.
(c) Construct a family of smooth functions

$$f_T : \mathbb{R}^n \to \mathbb{R}$$

over $T > 0$ such that the pointwise limit of $f_T|_{H^c}$ equals f.
(d) Can you elaborate on uniform convergence of f_T?
(e) Consider the gradient flow trajectories of f_T for $T > 0$, and discuss mathematically how they can help to find the minimum of f. For instance, how does the gradient vector field look like for T close to 0? What will happen if the initial point lies in a connected component where f is minimum? Formulate and prove as many mathematical statements as you can.