LINEAR ALGEBRA HOMEWORK 4

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Assume U, V, W are F-vector spaces. Put $\operatorname{End} V = \operatorname{Hom}(V, V)$.

Exercise 0.1. *Let $f, g : V \to V$ be two given maps such that $f \circ g = id_V$.*
(a) Show that g is injective and f is surjective.
(b) Assume in addition that $\dim V < +\infty$ and f is linear. Show that f is injective, hence g is surjective. (Hint: Use COD.)
(c) Conclude that g is bijective, and that $f = g^{-1}$ and $g \circ f = id_V$.
(d) Let $A, B \in M_{n,n}$. Show that if $AB = I$, then $BA = I$.
(e) Second proof. Show that if $\ker(BA) = (0)$ then $\ker A = (0)$, hence A is an isomorphism. Conclude that $B = A^{-1}$.
(Hint: COD.)

Exercise 0.2. *WRITE UP Let $V = F[t]$. In 3 lines show that there is a unique linear map $g : V \to V$ such that $g : t^n \mapsto t^{n+1}$ for $n \geq 0$. Likewise there is a unique linear map $f : V \to V$ such that $f : t^{n+1} \to t^n$ for $n \geq 0$, and $1 \mapsto 0$. Compute $f \circ g$ and $g \circ f$.*

Exercise 0.3. *WRITE UP Prove that for $A \in M_{n,n}$, $\det A^t = \det A$. You will need the fact that $\operatorname{sgn} \sigma^{-1} = \operatorname{sgn} \sigma$ for any bijection of $\{1, 2, .., n\}$.*

Exercise 0.4. *Decide if $A = [e_3, e_1+e_2, e_2] \in M_{3,3}$ is invertible. If so, compute A^{-1}. Here e_i are the standard unit vectors if F^3.*

Exercise 0.5. *Let $U \subset V$ be a subspace and $x \in \operatorname{End} V$ such that $xU \subset U$. In 5 lines, prove that there is a canonical map*

$$\bar{x} : V/U \to V/U, \quad v + U \mapsto xv + U.$$

That is check that this is well-defined. Show it satisfies the following: if $p(t) \in F[t]$, and $p(x) = 0$ in $\operatorname{End} V$ then $p(\bar{x}) = 0$ in $\operatorname{End} V/U$.